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# Membranes from monopole operators in ABJM theory: large angular momentum and M-theoretic $\text{AdS}_4/\text{CFT}_3$

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## Abstract

We consider states with large angular momentum to facilitate the study of the M-theory regime of the  $\text{AdS}_4/\text{CFT}_3$  correspondence. More precisely, we study the duality between M-theory on  $\text{AdS}_4 \times S^7/\mathbb{Z}_k$  and  $\mathcal{N} = 6$  supersymmetric Chern–Simons–matter (ABJM) theory with gauge group  $U(N) \times U(N)$  and level  $k$ , in the regime where  $k$  is of order one and  $N$  is large. In this regime the study of both sides of the duality is challenging: the lack of an explicit formulation of M-theory in  $\text{AdS}_4 \times S^7/\mathbb{Z}_k$  makes the gravity side difficult, while the CFT side is strongly coupled and the planar approximation is not applicable. In order to overcome these difficulties, we focus on states on the gravity side with large orbital angular momentum  $J \gg 1$  associated with a single plane of rotation in  $S^7$ . We then identify the corresponding operators in the CFT, thereby establishing the AdS/CFT dictionary in this large angular momentum sector. We show that there are natural approximation schemes on both sides of the correspondence as a consequence of the presence of the small parameter  $1/J$ . On the AdS side, the sector we focus on is well-approximated by the matrix model of M-theory – with matrices of size  $J/k$  – defined on the maximally supersymmetric eleven-dimensional pp-wave background. The pp-wave approximation to M-theory in  $\text{AdS}_4 \times S^7/\mathbb{Z}_k$  is justified for  $1 \ll J \ll N^{1/2}$ , while loop corrections in the matrix model are suppressed compared to tree-level contributions for  $J \gg N^{1/3}$ . On the CFT side, we study the ABJM theory defined on  $S^2 \times \mathbb{R}$  with large magnetic flux  $J/k$ . Using a carefully chosen gauge, we find an expansion of the Born–Oppenheimer type which arises naturally for large  $J$  in spite of the theory being strongly coupled. The energy spectra computed on the two sides agree at leading order. This provides a highly non-trivial test of the  $\text{AdS}_4/\text{CFT}_3$  correspondence including near-BPS observables associated with membrane degrees of freedom, therefore extending the validity of the  $\text{AdS}_4/\text{CFT}_3$  duality beyond the previously studied sectors corresponding to either BPS supergravity observables or the type IIA string regime.

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# 1 Introduction

Our understanding of non-perturbative aspects of string theory is still quite limited, although important progress has been made in recent years, thanks, in particular, to work on string dualities and D-branes. It is very important to consolidate and further this progress. M-theory [1,2], a conjectured eleven-dimensional theory which arises as strong coupling limit of type IIA string theory, plays a crucial role in this area. Various general features of M-theory are understood – it does not contain a dimensionless coupling constant (the only parameter in the theory is the eleven-dimensional Planck length), it reduces to eleven-dimensional supergravity in the low-energy limit and it contains among its excitations M2- and M5-branes, for which a classical action is known. These classical properties have many non-trivial consequences and implications for non-perturbative string theory. However, a well established formulation of M-theory in terms of its fundamental degrees of freedom is still lacking. In order to fully exploit the power of M-theory and elucidate its role in establishing a truly non-perturbative picture of string theory, it is crucial to develop a better understanding of the microscopic formulation of the theory including a consistent framework for its quantisation. The best candidate for such a formulation is currently the matrix model of M-theory.

In this paper we present a proposal for the study of a sector of M-theory combining the matrix model approach with the AdS/CFT correspondence. We show how the AdS/CFT duality can be studied in a genuinely M-theoretic regime by focussing on a particular set of states characterised by a large orbital angular momentum. Taking advantage of the dual description of these states in terms of a CFT allows us at the same time to independently confirm the results of the matrix model analysis. In this way, we simultaneously check the validity of both the matrix model proposal and the AdS/CFT correspondence.

The matrix model of M-theory can be considered as a regularised version of the theory describing (super)membrane degrees of freedom [3,4]<sup>1</sup>. In this approach the embedding coordinates of the membrane and their fermionic superpartners are replaced by  $K \times K$  matrices<sup>2</sup> and the resulting theory describes a quantum mechanical system with a finite number of degrees of freedom. The size of the matrices plays the role of a regulator and the quantum theory of the (super)membrane is expected to arise in the  $K \rightarrow \infty$  limit. The same matrix model is found in type IIA string theory as describing the low-energy dynamics of a system of D-particles (D0-branes) [5,6]. In this context the size of the matrices is associated with the number of D0-branes. In [6] it was conjectured that the  $K \rightarrow \infty$  limit of this supersymmetric

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<sup>1</sup>More precise statements are the following: (i) for a given regularisation parameter (the size of the matrices) and for a sufficiently smooth configuration in the membrane theory, which in general describes multiple membranes, there is a corresponding configuration in the matrix model; (ii) the classical action functionals for the configuration in the continuum membrane theory and that for the corresponding configuration in the discrete matrix model approximately match; (iii) the approximation becomes better, for a fixed configuration in the continuum theory, when the size of the matrices becomes larger, provided that the parameters of the discrete theory have the appropriate dependence on the regularisation parameter; this dependence defines the classical continuum limit. The above properties imply that the semi-classical approximation to the path integral of the matrix model includes contributions which are governed by a Boltzmann factor associated approximately with the action functional of the membrane theory. In this sense the matrix model contains (multi-)membranes. In order to have a complete understanding of this issue it is necessary to address questions such as “Does the matrix model contain other degrees of freedom such as M5-branes?” and “What should be the quantum continuum limit?”

<sup>2</sup>In the literature the size of the matrices in the matrix model is usually denoted by  $N$ . Here we use the letter  $K$  to avoid confusion with the parameter  $N$  used in the context of the AdS/CFT correspondence.

matrix model capture the entire dynamics of M-theory.

A complete and satisfactory understanding of the large  $K$  limit of the matrix model is still lacking and this represents a major obstacle in establishing it as a viable description of M-theory. Another unresolved issue concerns the emergence of the eleven-dimensional Lorentz symmetry [7–11]. No complete proof that a Lorentz invariant quantum theory arise in the large  $K$  limit is known. In particular the construction of the matrix model is closely tied to the use of light-front quantisation and no manifestly Lorentz-invariant formulation is known.

In order to substantiate the matrix model proposal it is necessary to address the fundamental issue of identifying proper observables in M-theory and then understanding how to realise them in the matrix model itself. Moreover a concrete scheme for the calculation of such observables should be identified and this is rendered challenging in particular by the absence of a dimensionless coupling constant. The majority of the tests of the matrix model approach to M-theory involve either the low-energy supergravity approximation or compactification to type IIA string theory in ten dimensions. Without considering such limits it is difficult to decide whether any results obtained from the matrix model are correct, although strong constraints should come from consistency requirements associated with unitarity and Lorentz invariance. In this paper we propose an approach which brings the AdS/CFT correspondence into the picture in order to overcome some of these limitations and make progress on these issues.

More specifically the use of the AdS/CFT dictionary allows us to identify quantities which are dual to CFT observables as “good” observables in the matrix model. Moreover, being able to independently compute such observables on the two sides of the duality, we are able to justify the results of the M-theory calculations. We will carry out this programme in a sector containing M2-brane states in M-theory, without resorting to a limit in which eleven dimensional supergravity or type IIA string theory can be used.

The specific AdS/CFT duality that we focus on in this paper, which we refer to as the  $\text{AdS}_4/\text{CFT}_3$  correspondence hereafter, was proposed in [12, 13]. It relates M-theory in an  $\text{AdS}_4 \times S^7/\mathbb{Z}_k$  background to a Chern-Simons-matter gauge theory with  $\mathcal{N} = 6$  supersymmetry. This theory, which we will refer to as the ABJM theory, has  $U(N) \times U(N)$  gauge group – with level  $k$  and  $-k$  for the two factors – and was first constructed in [13], following previous work [14–21]. It describes the low-energy limit of the dynamics of  $N$  coincident membranes in  $\mathbb{R}^8/\mathbb{Z}_k$ .

The  $\text{AdS}_4 \times S^7/\mathbb{Z}_k$  background arises as near-horizon geometry of such a stack of M2-branes. The  $\mathbb{Z}_k$  action is generated by  $2\pi/k$  rotations acting simultaneously in the 12, 34, 56, and 78 planes of  $\mathbb{R}^8$  in which the  $S^7$  is embedded. We denote the angular momentum generators associated with rotations in these four planes – which can be chosen as basis for the Cartan subalgebra of the  $SO(8)$  isometry group of  $S^7$  – by  $J_1, J_2, J_3$  and  $J_4$  respectively. The  $S^7$  can be described as an  $S^1$  fibration over  $\mathbb{CP}^3$ , where the  $S^1$  has constant radius and is generated at each point by  $J_M = J_1 + J_2 + J_3 + J_4$ . This is the  $S^1$  which is identified as the M-theory circle [13, 22] and the  $\mathbb{Z}_k$  quotient has the effect of dividing the circumference of this circle by  $k$ . For  $k \rightarrow \infty$  with  $N/k$  fixed the theory is compactified to ten dimensions and reduces to type IIA string theory in  $\text{AdS}_4 \times \mathbb{CP}^3$  [13]. This limit has been extensively studied after the original proposal [13] and corresponds to the ’t Hooft limit in the CFT, where  $N$  is large with  $\lambda = N/k$  fixed.

We are instead interested in studying a genuinely eleven-dimensional, M-theoretic, regime where  $k$  is of order 1 and  $N$  is large.

One reason to study the M-theory regime of the  $\text{AdS}_4/\text{CFT}_3$  duality is that one hopes to learn about M-theory in this way, as already discussed above. In particular, since the ABJM theory is conjectured to describe the low-energy dynamics of M2-branes, it is natural to ask whether there is a direct connection between this theory and the matrix model. One of the main results in this paper is to establish a natural and very direct connection between a certain sector of the ABJM theory and the pp-wave matrix model first formulated in [23].

Another motivation for our work comes from the possibility of gaining new insights into fundamental aspects of the AdS/CFT correspondence by studying it in a regime which is essentially different from what has been considered before. Although the AdS/CFT duality has been extensively studied, especially in its canonical version relating the  $\mathcal{N} = 4$  supersymmetric Yang–Mills (SYM) theory in four dimensions to type IIB string theory in an  $\text{AdS}_5 \times S^5$  background, important open questions remain concerning its foundations. In particular the fundamental mechanism underlying the correspondence is not fully understood. Analysing a non-stringy AdS/CFT, of which the M-theoretic regime of the  $\text{AdS}_4/\text{CFT}_3$  duality is a prime example, should help to shed light on this aspect, as certainly in this case the correspondence cannot be explained in terms of open/closed string duality. Another important feature of the regime we focus on is that it is not compatible with the use of the planar approximation, since it requires large  $N$  but  $k \sim 1$ , so that  $\lambda = N/k$  cannot be fixed. This is natural as the 't Hooft expansion suggests that the gauge theory should have a description in terms of string-like degrees of freedom, which is not the case in the M-theory regime. Therefore the sector we consider allows us to analyse the gauge/gravity duality independently of the special role played by the planar approximation.

The duality in the M-theoretic regime is considered to be rather non-tractable. On the CFT side, the theory is strongly coupled as  $k \sim 1$ . Furthermore, one cannot focus on the planar diagrams and all non-planar contributions are relevant. On the AdS side, one has to face the problem of formulating M-theory in  $\text{AdS}_4 \times S^7$ , in particular when trying to calculate observables including quantum corrections.

In this paper, we present evidence that when one introduces a large orbital angular momentum,  $J$ , the presence of the small parameter  $1/J$  makes it possible to identify good approximation schemes on both the CFT and the AdS sides. We construct the relevant observables on both sides and establish a dictionary between them. The spectra computed on the two sides match, verifying the AdS/CFT conjecture in an M-theoretic regime.

The idea of using a large angular momentum to obtain a workable approximation is natural as the WKB approach is usually applicable in cases where one has large quantum numbers (in our case  $J$ ). In the  $\text{AdS}_5/\text{CFT}_4$  context this idea has been put forward by Berenstein, Maldacena and Nastase (BMN) in [23], where it was shown that focussing on a large angular momentum sector leads to a situation in which both sides of the duality are weakly coupled and the AdS/CFT correspondence is directly testable. Our work is in many ways analogous to the BMN analysis, although with some important differences. We construct operators in the ABJM theory, which play a role analogous to the BMN operators. The construction of such operators is, however, totally different and this reflects the fact that they correspond to excited states of membranes rather than strings.

On the gravity side of the correspondence we describe the physics of states in  $\text{AdS}_4 \times S^7/\mathbb{Z}_k$

which belong to a sector characterised by large angular momentum. M-theory states are classified by the eigenvalues of the Cartan generators  $J_1, J_2, J_3$  and  $J_4$ . We focus on states which have large  $J_4$  and the other components of the angular momentum of order one. The dynamics of such states can be described using the maximally supersymmetric eleven dimensional pp-wave geometry to approximate the  $\text{AdS}_4 \times S^7/\mathbb{Z}_k$  background. Following the proposal to use the matrix model as a microscopic formulation of M-theory, it is then natural to adopt as framework for our calculations the pp-wave matrix model [23]. An important aspect of our proposal is that the size of the matrices in this matrix model should be identified with  $J_M/k$ .

The possible vacuum states in the large angular momentum sector are the BPS states of the pp-wave matrix model, which were studied in [24, 25]. The simplest such state is a fuzzy sphere configuration corresponding to a spherical membrane which extends in the  $\text{AdS}_4$  directions and is point-like in  $S^7$ , where it moves along a great circle with large angular momentum  $J$ . In general the BPS states correspond to a collection of concentric fuzzy spheres, labelled by a set of integers corresponding to the portion of the total angular momentum carried by the individual membranes. The radius,  $r$ , of the fuzzy spheres is proportional to  $J$ . The use of the pp-wave approximation is justified if this  $r$  is much smaller than the radius of curvature of the  $\text{AdS}_4$  and  $S^7$  factors in the original background. This leads to the condition

$$1 \ll J \ll N^{1/2} \tag{1.1}$$

for the applicability of the pp-wave approximation.

After identifying the ground state in the large  $J$  sector, we study the spectrum of fluctuations around the classical vacuum configurations. We present the tree level spectrum, which is determined by the pp-wave matrix model Hamiltonian at quadratic order in the fluctuations. We then discuss the behaviour of quantum corrections associated with cubic and quartic terms in the fluctuations. The condition that one-loop effects produce small corrections to the tree level result turns out to be

$$J \gg N^{1/3}, \tag{1.2}$$

using the results of [24]. It is crucial for our analysis that both conditions, (1.1) and (1.2), can be satisfied for large  $N$  choosing the parameter  $J$  so that

$$N^{1/3} \ll J \ll N^{1/2} \quad \text{i.e.} \quad J^2 \ll N \ll J^3. \tag{1.3}$$

Having analysed the large angular momentum sector of the pp-wave matrix model, we then describe the dual large  $J$  observables on the CFT side. These are gauge-invariant operators in the ABJM theory with quantum numbers matching those of the membrane states we discussed. We show that the requirement of gauge invariance leads to identify monopole operators as dual to membrane states in the large  $J$  sector. Monopole operators [26], which play a crucial role in the ABJM theory and also in three dimensional gauge theory in general [13, 27, 28], are classified by a set of integers, the so-called GNO charges [29], which satisfy a Dirac quantisation condition. We show, by focussing on BPS or ground states, that it is possible to identify the GNO charges of the relevant CFT operators with the angular momenta of the dual membrane states associated with motion along the great circle in  $S^7$ .



Monopole operators are associated with a Dirac monopole singularity at the insertion point. As such they do not have a simple manifestly local description in terms of the elementary fields in the theory. In order to deal with this complication it is convenient to use radial quantisation and study the ABJM theory on  $S^2 \times \mathbb{R}$  in Hamiltonian formulation in the presence of magnetic flux through the  $S^2$ . Using the state-operator map we identify the states in the radially quantised ABJM theory – in a sector characterised by large magnetic flux,  $J$  – which are dual to membrane excitations in the bulk. An important ingredient in this construction is the identification of a suitable gauge.

In this framework the dictionary relating the gravity and gauge sides arises in a natural way, leading to a very direct correspondence. Bulk states corresponding to spherical membranes and their excitations have a dual description in terms of states of the ABJM theory on  $S^2$ . Therefore states on the two sides of the duality are described in terms of the same spherical harmonics on  $S^2$ . The energy spectrum of the membrane excitations, which are in general non-BPS, corresponds to the energy spectrum of the ABJM theory in radial quantisation.

In the case where the ground state on the gravity side is a single membrane, we verify that the tree-level spectrum obtained from the matrix model calculation agrees with the leading order result on the CFT side for all types of bosonic and fermionic excitations.

Despite the fact that the ABJM theory is strongly coupled for  $k \sim 1$ , we argue that a perturbative expansion is possible using a Born-Oppenheimer type approximation. The presence of a large magnetic flux,  $J$ , induces a separation of energy scales which leads to a natural identification of slow (or low-energy) modes and fast (or high-energy) modes. Integrating out the fast modes leads to an effective low-energy Hamiltonian for the slow modes which is weakly coupled for large  $J$ . We propose that this approach provides a framework for the systematic study of quantum corrections in the ABJM theory in the large  $J$  sector that we defined.

We also discuss the generalisation to the case in which the ground state contains multiple membranes. The dual CFT sector involves monopole operators characterised by multiple non-zero GNO charges, corresponding to the angular momenta of the individual membranes. In this case the pp-wave matrix model vacuum consists of block-diagonal matrices. The excited states built on such vacua involve fluctuations in off-diagonal blocks, which do not correspond to degrees of freedom associated with individual membranes in the continuum. We will identify the dual states in the ABJM theory and show that in some cases – specifically when there are two membranes of approximately the same size and hence close to each other – these extra degrees of freedom on the two sides of the correspondence can be compared reliably and quantitatively within the limits of validity of our approximations. The agreement between the corresponding spectra is a strong indication that these states describe true degrees of freedom of M-theory, which are captured by the matrix model, but are not present in the conventional continuum membrane theory.

This paper is organised as follows. In section 2 we describe the AdS side of the correspondence. We discuss the pp-wave approximation for membranes in  $\text{AdS}_4 \times S^7/\mathbb{Z}_k$  and present the associated matrix model and its energy spectrum. In section 3 we describe the CFT side. We first discuss the Hamiltonian formulation of the ABJM theory in  $S^2 \times \mathbb{R}$ . We then explain the separation between fast and slow modes in the framework of the Born-Oppenheimer approximation and present the energy spectrum in the large  $J$  sector. Particular attention

is devoted to the discussion of gauge-fixing which plays an essential role in our analysis. In the discussion of both sides of the duality we first consider BPS states (ground states) and then near-BPS states (fluctuations around the ground state), which are not protected and receive quantum corrections. In section 4 we discuss the case of multi-membrane vacua. We conclude in section 5 with a discussion of our result and an outline of possible extensions and generalisations.

## 2 AdS side

In this section, we describe the AdS side of the correspondence. We begin by recalling some basic formulae in M-theory and the  $\text{AdS}_4/\text{CFT}_3$  duality.

M-theory has only one length scale and the membrane tension,  $T$ , is directly related to the eleven dimensional Planck length. We use the conventions of [12, 13] in which the Planck length is defined so that the Einstein-Hilbert part of the  $D = 11$  supergravity action reads

$$S = -\frac{1}{2^8 \pi^8 l_P^9} \int d^{11}x \sqrt{-g} \mathcal{R} + \dots \quad (2.1)$$

The relation between the membrane tension and the Planck length is then [30]

$$T = \frac{1}{4\pi^2 l_P^3}. \quad (2.2)$$

The  $\text{AdS}_4/\text{CFT}_3$  correspondence proposed in [12, 13] was constructed considering the near horizon limit of a stack of  $N$  M2-branes in  $\mathbb{R}^8/\mathbb{Z}_k$  (which may be understood as a certain projection of  $Nk$  M2-branes in flat space). The resulting geometry is  $\text{AdS}_4 \times S^7/\mathbb{Z}_k$ , where the radius of the  $S^7$ ,  $R$ , in terms of the eleven-dimensional Planck length satisfies

$$2^5 \pi^2 Nk = \frac{R^6}{l_P^6}, \quad (2.3)$$

while the radius of curvature of the  $\text{AdS}_4$  factor is

$$R' = \frac{1}{2}R. \quad (2.4)$$

We shall now specify the kinematical regime we study in this paper. Corresponding to rotations in the 12, 34, 56 and 78 planes of  $\mathbb{R}^8$  in which the  $S^7$  is embedded, there are four angular momentum quantum numbers,  $J_1, J_2, J_3$  and  $J_4$ . The states we focus on are those for which one of them, which conventionally we take to be  $J_4$ , is large and the other angular momentum quantum numbers are of order 1.

Another important quantum number is  $J_M = J_1 + J_2 + J_3 + J_4$ . This is related to the momentum along the M-theory circle, which in the  $\text{AdS}_4 \times S^7/\mathbb{Z}_k$  background is identified with the great circle (or rather the family of great circles) corresponding to the orbit of the  $J_M$  generator<sup>3</sup> [13, 22]. The states we are interested in have  $J_4 \gg 1$ ,  $J_M \gg 1$  and  $J_M - J_4 \sim 1$ . Since we focus on the leading order terms in a  $1/J$  expansion, the difference between  $J_4$  and  $J_M$  will often be irrelevant. Hence in most instances we will simply write  $J$  instead of  $J_4$  or

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<sup>3</sup>For points in the 78 plane, the M-theory circle coincides with the equator generated by  $J_4$  rotations.



$J_M$ . When the distinction between the two is relevant, we will explicitly specify whether we are referring to  $J_4$  or  $J_M$ .

In the following we first consider  $k = 1$  and then generalise to the case of  $k \neq 1$ , which is obtained by a certain projection. Since the  $\mathbb{Z}_k$  quotient acts on the M-theory circle, the projection requires the  $J_M$  quantum number of any individual object to be a multiple of  $k$  (while of course  $J_4$  can take any integer value). In section 4 we discuss configurations containing multiple membranes carrying large momentum along the M-theory circle. In that case the  $\mathbb{Z}_k$  quotient implies that the  $J_M$  of each individual membrane should be a multiple of  $k$ .

The dynamics of objects (both point-like and extended, such as strings or membranes) propagating in a curved geometry with large spatial momentum can be described using an approximation scheme referred to as the pp-wave approximation [23,31]. This can be understood as an extension of the familiar infinite momentum frame argument (or the ultra-relativistic limit) in flat space to the case of a curved background. As is well known, the dynamics of an object having very large spatial momentum in flat space-time is approximately governed by a free non-relativistic Hamiltonian. If the background space-time is curved, the dynamics of objects with very large spatial momentum, proportional to a parameter  $J$ , is instead approximately controlled by a non-relativistic Hamiltonian containing an external harmonic oscillator potential, whose strength is determined by the curvature radius and by  $J$ .

The same Hamiltonian with a suitable identification of parameters also describes the dynamics of objects in a so-called pp-wave geometry. There is a limiting process, referred to as Penrose limit [32–34], which produces the pp-wave geometry starting from the original background. However, we stress that the point of view that we take in this paper is to treat the procedure as an approximation scheme to describe the dynamics of special states in the AdS/CFT correspondence. Rather than viewing the pp-wave background as arising from a formal limit between two geometries, we consider it as an approximation which allows to capture the dynamics of states with large spatial momentum propagating in the original space-time [31].

Let us recall the essential points of the pp-wave approximation by using a simple example, a massless particle in the space-time  $\mathbb{R} \times S^n$  with metric

$$ds^2 = -(dx^0)^2 + R^2 d\Omega_n^2, \quad (2.5)$$

where  $d\Omega_n^2$  is the line element on the  $n$ -dimensional unit sphere  $S^n$  with  $n \geq 2$ . The dynamics of the particle is governed by the mass shell condition

$$g^{ij} P_i P_j = 0. \quad (2.6)$$

We temporarily use the indices  $i, j = 0, \dots, n$  to label the coordinates of the space-time (2.5). We focus on a great circle in  $S^n$ . We then assume that the particle has large momentum along this fixed circle and does not deviate far from it. Let the spatial coordinate  $x^1$  be defined as the angle around the fixed large circle multiplied by the radius  $R$ . The longitudinal momentum  $P_1 > 0$  conjugate to  $x^1$  is by assumption large. We choose the transverse coordinates  $x^\alpha$ ,  $\alpha = 2, \dots, n$ , in the directions orthogonal to the great circle. In terms of these coordinates the metric is approximately

$$ds^2 \approx -(dx^0)^2 + \left(1 - \frac{(x^\alpha)^2}{R^2}\right) (dx^1)^2 + (dx^\alpha)^2, \quad (2.7)$$

neglecting higher order terms in  $x^\alpha/R$ . Using (2.7), the dispersion relation (2.6) becomes

$$(-P_0)^2 \approx \left(1 + \frac{(x^\alpha)^2}{R^2}\right) (P_1)^2 + (P_\alpha)^2. \quad (2.8)$$

Large longitudinal momentum  $P_1$  therefore implies large energy  $-P_0 > 0$ . The finite difference is given by

$$(-P_0) - P_1 \approx \frac{(P_\alpha)^2 + (P_1)^2 \frac{(x^\alpha)^2}{R^2}}{2P_1}. \quad (2.9)$$

where we used  $(-P_0) + P_1 \approx 2P_1$ . Equation (2.9) shows that, for fixed (large) longitudinal momentum,  $P_1$ , the dynamics of the particle in curved space is approximately that of a non-relativistic harmonic oscillator. Notice that the longitudinal momentum is actually quantised  $-P_1 = J/R$ , where  $J$  is an integer – because its conjugate coordinate  $x^1$  is periodic with period  $2\pi R$ . Therefore in this approximation

$$(-P_0) + P_1 \approx 2P_1 = 2 \frac{J}{R}. \quad (2.10)$$

Equation (2.7) is valid if  $|x^\alpha|/R \ll 1$ . Classically one can assume a particle to remain arbitrarily close to the fixed great circle. However, in the quantum theory the wave function of the particle has finite extension. For the  $n$ -th excited state, the extension can be estimated using (2.9) and  $P_1 = J/R$ ,

$$\langle x \rangle \sim R \sqrt{\frac{2n+1}{J}}. \quad (2.11)$$

Hence we see that the condition  $\langle x \rangle \ll R$ , which validates the use of the pp-wave approximation, gives an upper bound on the excitation number,

$$n \ll J, \quad (2.12)$$

and also implies

$$J \gg 1. \quad (2.13)$$

Another way of understanding the above formulae is in terms of a centrifugal potential. Because of the large angular momentum, the particle experiences a strong centrifugal force confining it around the equator (where the radius of the trajectory is the largest – the centrifugal force pushes objects in the direction where the radius becomes larger). The pp-wave approximation keeps the leading order term in this centrifugal potential, which as expected has the harmonic oscillator form. The strength of the potential is determined by the curvature radius of the background and the (angular) momentum.

The use of the pp-wave approximation in the context of the AdS/CFT correspondence involves an additional subtlety. In order to have a consistent dictionary between the gravity and CFT sides, it is necessary to change the space-time picture on the AdS side to the one given in [35, 36] which is particularly suited for studying holographic aspects (*i.e.* the computation of correlation functions following the prescription in [37, 38]). More specifically, one should not consider objects (particles, strings or membranes) propagating in the AdS space (with oscillating wave functions), but rather one should consider objects undergoing a tunnelling process (with exponentially decreasing or increasing wave functions). In practice this is achieved by a certain double Wick rotation. This prescription was proposed

in [35, 36] for the pp-wave approximation to string theory in  $\text{AdS}_5 \times S^5$ . It solves various puzzles regarding the signature of the bulk/boundary, including the identification of energy and conformal dimension and the signature of vector type fluctuations. Although the new interpretation is different leading to a better, consistent correspondence, the end result of the pp-wave approximation is mathematically equivalent [35]. Both of the interpretations, with or without the double Wick rotation, lead to the same effective Hamiltonian in the pp-wave approximation. This is the case even for more general backgrounds corresponding to near horizon limits of  $Dp$ -brane configurations [39]. The same interpretation has also been applied to the computation of correlation functions using methods derived from the study of integrable systems in [40, 41]. We shall not elaborate on this issue any further and we refer the reader to [35, 36] for additional details. In the following we assume that the identification of observables between the gravity and CFT sides of the correspondence is made adopting the prescription discussed in these papers.

Applying the above considerations to the study of M-theory in  $\text{AdS}_4 \times S^7/\mathbb{Z}_k$ , we conclude that the dynamics of states with large angular momentum in this background can be described using a suitable pp-wave approximation. Combining this idea with the matrix model proposal leads us naturally to use a matrix model which has the same form as the one arising in the maximally supersymmetric eleven-dimensional pp-wave geometry [34, 42]. This matrix model was first proposed in [23] and it was later derived in [24, 43] from the regularisation of the supermembrane theory in the pp-wave background. This matrix model is the main ingredient in our analysis of the gravity side of the  $\text{AdS}_4/\text{CFT}_3$  duality.

In the spirit of using the pp-wave background as an approximation scheme to study a large angular momentum sector of M-theory in  $\text{AdS}_4 \times S^7$ , we will write the matrix model in terms of parameters characterising the original geometry, *i.e.* the radii  $R$  and  $R' = R/2$ , and the angular momentum parameter  $J$ <sup>4</sup>. We first consider the membrane theory in  $\text{AdS}_4 \times S^7/\mathbb{Z}_k$  in the pp-wave approximation and then regularise it to obtain the matrix model. Rather than providing a detailed derivation of the membrane Hamiltonian starting from the supermembrane theory in  $\text{AdS}_4 \times S^7/\mathbb{Z}_k$  (analogous to that in [31, 35] for the type IIB string in the  $\text{AdS}_5 \times S^5$  background), we will justify its form based on the same arguments that led to (2.9). The physics of membranes in  $\text{AdS}_4 \times S^7/\mathbb{Z}_k$  can be captured by simply restricting the attention to those special states of the supermembrane theory in  $\text{AdS}_4 \times S^7$  [44] for which all individual membranes have a  $J_M$  quantum number which is a multiple of  $k$ .

The bosonic part of the membrane Hamiltonian<sup>5</sup> in the pp-wave approximation is

$$\begin{aligned}
-P_0 - P_1 = & \int d^2\sigma \left( \frac{[\sigma]}{2P_1} (p_\alpha)^2 + \frac{[\sigma]}{2P_1} \frac{1}{2} T^2 (\{x^m, x^n\}^2 + \{y^i, y^j\}^2 + 2\{x^m, y^i\}^2) \right. \\
& \left. + \frac{1}{2} \frac{P_1}{[\sigma]} \frac{(x^m)^2}{R^2} + \frac{1}{2} \frac{P_1}{[\sigma]} \frac{(y^i)^2}{(R')^2} - \frac{T}{2R'} \epsilon_{ijk} y^i \{y^j, y^k\} \right), \quad (2.14)
\end{aligned}$$

where  $T$  is the membrane tension (2.2) and the nine transverse coordinates have been denoted by  $x$  and  $y$ , with  $y^i$ ,  $i = 1, 2, 3$ , indicating three scalars originating from  $\text{AdS}_4$  directions and  $x^m$ ,  $m = 4, \dots, 9$ , referring to six scalars originating from  $S^7$  directions. We also use

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<sup>4</sup>In particular we do not introduce a mass parameter,  $\mu$ , as commonly done in the literature. This introduction of  $\mu$  is not necessary for the comparison between observables on the AdS side and the CFT side and moreover it makes the analysis of the limits of validity of the pp-wave approximation less transparent.

<sup>5</sup>We will often refer to the combination  $-P_0 - P_1$  as the Hamiltonian on the AdS side of the correspondence.

$\alpha = 1, \dots, 9$  to refer to the set of all nine transverse directions. In the following we will use the notation  $x^\alpha$  to collectively denote all the membrane coordinates when we do not need to distinguish between  $\text{AdS}_4$  and  $S^7$  directions. The Lie bracket,  $\{.,.\}$ , in (2.14) is defined as

$$\{f, g\} = \frac{\partial f}{\partial \sigma_1} \frac{\partial g}{\partial \sigma_2} - \frac{\partial f}{\partial \sigma_2} \frac{\partial g}{\partial \sigma_1}, \quad (2.15)$$

for any functions,  $f(\sigma_1, \sigma_2)$  and  $g(\sigma_1, \sigma_2)$ , on the membrane world-volume. The constant  $[\sigma]$  is the total area of the base space,

$$[\sigma] = \int d^2\sigma. \quad (2.16)$$

It should not of course appear in observable quantities and we will see later that  $[\sigma]$  does not appear after the regularisation.  $P_1$  is the momentum along the equator of the  $S^7$ . It is related to the (integer-valued) quantum number  $J$  by

$$P_1 = \frac{J}{R}, \quad (2.17)$$

where to be precise  $J$  in the numerator should be understood as the value of  $J_4$ .  $-P_0 > 0$  is a similar quantity associated with a “time-like” direction in  $\text{AdS}_4$ , which, by the conventional dictionary of the  $\text{AdS}/\text{CFT}$  duality, is related to the conformal dimension  $\Delta$  of the dual CFT operators by <sup>6</sup>

$$-P_0 = \frac{\Delta}{R'} = 2 \frac{\Delta}{R}. \quad (2.18)$$

The various terms in (2.14) can be understood as follows. The quadratic terms in the  $x$  and  $y$  coordinates come from the harmonic oscillator potential arising in the pp-wave approximation, analogous to the quadratic term appearing in (2.9). The cubic term for the  $y$ ’s is induced by the coupling of the membrane to the three-form potential, which has non-zero background value in the  $\text{AdS}_4$  space. The remaining terms are those appearing in the membrane Hamiltonian in flat space in the ultra-relativistic limit <sup>7</sup>. We have fixed partially the reparametrisation invariance of the membrane in a way analogous to that used in the light-cone gauge for membranes in flat space-time [3, 4]. The Hamiltonian (2.14) can be rewritten in the form of a sum of squares, which simplifies the study of the minima of the potential,

$$\begin{aligned} -P_0 - P_1 = \int d^2\sigma \left( \frac{[\sigma]}{2P_1} (p_\alpha)^2 + \frac{[\sigma]}{2P_1} \frac{1}{2} T^2 (\{x^m, x^n\}^2 + 2\{x^m, y^i\}^2) + \frac{1}{2} \frac{P_1}{[\sigma]} \frac{(x^m)^2}{R^2} \right. \\ \left. + \frac{[\sigma]}{2P_1} \left( \frac{1}{2} T \epsilon_{ijk} \{y^j, y^k\} - \frac{P_1}{[\sigma]} \frac{y^i}{R'} \right)^2 \right). \end{aligned} \quad (2.19)$$

There is also the phase space constraint

$$\{x^\alpha, p^\alpha\} = 0, \quad (2.20)$$

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<sup>6</sup>The identification becomes quite direct and transparent in the interpretation discussed in [35, 36].

<sup>7</sup>Those familiar with the light-cone gauge formulation of the membrane theory might wonder whether we are working in the light-cone gauge or using the ultra-relativistic limit (also called the infinite momentum frame in the case of flat space-time). Arguably, it makes sense to distinguish the two points of view in flat space since the light-cone gauge gives exact results and it is applicable to generic states, whereas the ultra-relativistic limit is an approximation valid only for special states. However, this distinction is meaningless in the present case of a curved space-time in which we have to make an approximation – the pp-wave approximation – and consider special states with large angular momenta.

associated with the residual gauge symmetry corresponding to the area preserving diffeomorphisms.

The matrix model which we will use in the following is obtained by regularising the Hamiltonian described in the previous paragraphs. An essential element of our proposal is that the proper matrix model regularisation, suitable to describe the large  $J$  sector of the  $\text{AdS}_4/\text{CFT}_3$  duality, should use matrices of size  $K = J_M/k$ .

One way to understand this identification is to notice that the D0-brane charge, which should be the matrix size [6], is equal to  $J_M/k$ . This follows from the identification of the M-theory circle in the  $\text{AdS}_4/\text{CFT}_3$  correspondence with the orbits of  $J_M$  acting on the  $\text{AdS}_4 \times S^7/\mathbb{Z}_k$  space-time.

Another way to understand the identification of the matrix size with the angular momentum, which is based on the interpretation of the matrix model as regularised membrane theory, is the following. In our gauge fixing of the membrane theory, we choose the space-like coordinates on the world-volume so that the longitudinal momentum density is constant on a time-slice of the world-volume. This implies that the (base-space) area of a certain portion of the time-slice of the world-volume is proportional to the longitudinal momentum contained in that portion. The longitudinal momentum is approximately equal to the momentum along the M-theory circle to leading order in our approximation. Because of the periodicity of the angle along the M-theory circle, the associated momentum has a minimum,  $k/R$ . This minimum of the momentum implies a minimum for the area in the time-slice of the world-volume of the membrane. The total area is proportional to the total momentum,  $J/R$ , and the minimum of the area is proportional to  $k/R$  with the same coefficient of proportionality. Hence the time-slice of the world-volume is divided into  $J/k$  pieces. This is achieved by regularising the membrane world-space by matrices, as the matrix regularisation corresponds to dividing the membrane world-space into equal area pieces. The number of these pieces is equal to the matrix size  $K = J/k$ . This can be understood using an analogy with the quantisation of a system with a single degree of freedom: the Bohr-Sommerfeld quantisation says that the minimum area of the phase space (the membrane world-space) is quantised in units of  $2\pi\hbar$  which is equal to the total area divided by the matrix size in the membrane context. Therefore the size of the matrices should be  $J/k$ <sup>8</sup>. The use of finite dimensional matrices in the presence of the compact longitudinal direction is reminiscent to the DLCQ argument presented in [45].

Let us recall some basic relations used in the matrix regularisation. A comprehensive review can be found in [46]. In this paper we follow the conventions of [47]. Functions on the membrane world-volume at fixed time,  $f(\sigma^1, \sigma^2)$ ,  $g(\sigma^1, \sigma^2), \dots$ , are replaced by  $K \times K$  matrices,  $\hat{f} = \rho(f)$ ,  $\hat{g} = \rho(g), \dots$ , where the map  $\rho$  is linear. These matrices provide a discrete approximation to the corresponding functions. The basic operations on functions have counterparts on the associated matrices. This correspondence can be summarised as follows,

$$\rho(fg) \approx \frac{1}{2} \left( \rho(f)\rho(g) + \rho(g)\rho(f) \right), \quad (2.21)$$

$$\rho(\{f, g\}) \approx \frac{2\pi K}{i[\sigma]} [\rho(f), \rho(g)] , \quad (2.22)$$

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<sup>8</sup> A similar interpretation can be applied to the case of the BMN analysis of the  $\text{AdS}_5/\text{CFT}_4$  duality. In that case fixed-time slices of the string world-sheet are discretised to a lattice with  $J$  sites, conforming with the construction of BMN operators on the CFT side.

$$\frac{1}{[\sigma]} \int f d^2\sigma \approx \frac{1}{K} \text{tr}(\rho(f)). \quad (2.23)$$

The symbol  $\approx$  indicates that the two sides of these relations are equal up to higher order corrections in  $1/K$ . The first equation simply states that the product of two functions becomes the multiplication (or more precisely one half the anti-commutator) of the corresponding matrices. The second equation relates the Lie bracket of two functions to the commutator of the associated matrices multiplied by a factor proportional to  $K$ .

Following this procedure, we introduce the matrix version of the membrane coordinates,

$$X^m = \rho(x^m), \quad Y^i = \rho(y^i). \quad (2.24)$$

The canonical conjugates of these matrices,  $P_\alpha$ , are related to the matrix version of the continuum momentum,  $p_\alpha$ , by

$$P_\alpha = \frac{[\sigma]}{K} \rho(p_\alpha). \quad (2.25)$$

Using  $K = J/k$ , the complete matrix model Hamiltonian takes the form

$$\begin{aligned} -P_0 - P_1 = \text{tr} \Bigg\{ & \frac{R}{2k} (P_\alpha)^2 - (2\pi T)^2 \frac{R}{2k} \frac{1}{2} ([X^m, X^n]^2 + 2[X^m, Y^i]^2 + [Y^i, Y^j]^2) \\ & + \frac{k}{2R^3} (X^m)^2 + \frac{k}{2RR^2} (Y^i)^2 + i2\pi T \frac{1}{R} \epsilon_{ijk} Y^i [Y^j, Y^k] \\ & + 2\pi T \frac{R}{k} \frac{1}{2} (\Psi^T \gamma^m [X^m, \Psi] + \Psi^T \gamma^i [Y^i, \Psi]) - \frac{3i}{4} \frac{1}{R} \Psi^T \gamma^{123} \Psi \Bigg\}, \end{aligned} \quad (2.26)$$

where we have also written the fermionic terms which were omitted in the membrane theory. Here  $\gamma^\alpha$  are SO(9) gamma-matrices and  $\gamma^{123} = \gamma^1 \gamma^2 \gamma^3$ . As in the case of the membrane Hamiltonian, the bosonic part of (2.26) can be rewritten as a sum of squares,

$$\begin{aligned} -P_0 - P_1 = \text{tr} \Bigg\{ & \frac{R}{2k} (P_\alpha)^2 - (2\pi T)^2 \frac{R}{2k} \frac{1}{2} ([X^m, X^n]^2 + 2[X^m, Y^i]^2) \\ & + \frac{k}{2R^3} (X^m)^2 - (2\pi T)^2 \frac{R}{2k} \left( \frac{1}{2} \epsilon_{ijk} [Y^j, Y^k] - i \frac{1}{2\pi T} \frac{2k}{R^2} Y^i \right)^2 \\ & + 2\pi T \frac{R}{k} \frac{1}{2} (\Psi^T \gamma^m [X^m, \Psi] + \Psi^T \gamma^i [Y^i, \Psi]) - \frac{3i}{4} \frac{1}{R} \Psi^T \gamma^{123} \Psi \Bigg\}, \end{aligned} \quad (2.27)$$

where we used  $R' = R/2$ .

We note that the use of  $K = J_M/k$  implies that the M-theory charge  $J_M$  should be a multiple of  $k$  for any object in the matrix model, since the size of the matrices is of course an integer. This property justifies our proposal that the matrix model describes physics on  $\text{AdS}_4 \times S^7/\mathbb{Z}_k$ , rather than  $\text{AdS}_4 \times S^7$ , in the pp-wave approximation.

The canonical (anti-)commutation relations are

$$[X^{\alpha r}_s, P^{\beta u}_v] = i\delta^{\alpha\beta} \delta^r_v \delta^u_s, \quad (2.28)$$

$$[\Psi^{ar}_s, \Psi^{bu}_v]_+ = \delta^{ab} \delta^r_v \delta^u_s, \quad (2.29)$$

where  $X^\alpha$ ,  $\alpha = 1, \dots, 9$ , collectively denotes the matrices associated with all nine membrane coordinates,  $a, b = 1, \dots, 16$  are SO(9) Majorana spinor indices and  $r, s, u, v = 1, \dots, K$  are matrix indices.

The phase space constraints are

$$[X^\alpha, P^\alpha] - i\Psi^T \Psi = 0, \quad (2.30)$$

where again the sum over  $\alpha$  runs from 1 to 9.

## 2.1 BPS states

Classical stable solutions of the pp-wave matrix model with zero energy are known [24]. They are the BPS states (or the ground states) in the sector we are studying in this paper. They are given by a collection of so-called fuzzy spheres extending in the 3 transverse directions originating from AdS<sub>4</sub> and are point-like in the  $S^7$  directions.

From the form of the matrix model Hamiltonian (2.27), which is written as a sum of squares, it is clear that minimum energy configurations have  $X^m = 0$  for  $m = 4, \dots, 9$ , so that the only non-vanishing fields in the classical solution are  $Y_0^i$ ,  $i = 1, 2, 3$ . They should satisfy

$$\frac{1}{2}\epsilon_{ijk}[Y_0^j, Y_0^k] - i\frac{1}{2\pi T}\frac{2k}{R^2}Y_0^i = 0. \quad (2.31)$$

This equation is solved by taking the  $Y_0^i$ 's to be proportional to  $K \times K$  generators,  $L^i$ , of a representation of SU(2). The explicit form of the solution is

$$Y_0^i = \frac{2k}{(2\pi T)R^2} L^i, \quad (2.32)$$

where the  $L^i$ 's obey

$$[L^i, L^j] = i\epsilon^{ijk}L^k. \quad (2.33)$$

The simplest solution corresponds to choosing the  $L^i$ 's to be the generators of the irreducible  $K$  dimensional SU(2) representation. Taking the proportionality constant so that  $Y_0^i$  is written as

$$Y_0^i = r\sqrt{\frac{4}{K^2 - 1}} L^i, \quad (2.34)$$

one finds for the parameter  $r$

$$r = \frac{k\sqrt{K^2 - 1}}{2\pi TR^2} \approx \frac{J}{2\pi TR^2}, \quad (2.35)$$

where we used  $K = J/k$  and  $J \gg 1$ . Equations (2.34) and (2.35) have then a simple geometric interpretation. A spherical membrane of unit radius, described by coordinates  $y^i$ ,  $i = 1, 2, 3$ , with  $\sum (y^i)^2 = 1$ , is approximated in the matrix model (with matrices of size  $K$ ) by the configuration  $Y^i = \sqrt{4/(K^2 - 1)} L^i$ , referred to as a fuzzy sphere of unit radius [3, 4]. Therefore the solution (2.34) corresponds to a fuzzy sphere of radius  $r$  given by (2.35).

A more general solution to (2.31) can be obtained considering a reducible  $K$  dimensional representation of SU(2). Equation (2.31) can be satisfied taking the  $Y$ 's to be block-diagonal



matrices,

$$Y_0^j = \begin{bmatrix} Y_{0(1)}^j & & & \\ & \ddots & & \\ & & Y_{0(i)}^j & \\ & & & \ddots \end{bmatrix}, \quad (2.36)$$

where the  $i$ -th block on the diagonal,  $Y_{0(i)}^j$ ,  $i = 1, \dots, n$ , is of size  $K_{(i)} = J_{(i)}/k$ . It is given by

$$Y_{0(i)}^j = r_{(i)} \sqrt{\frac{4}{K_{(i)}^2 - 1}} L_{(i)}^j, \quad (2.37)$$

where

$$r_{(i)} = \frac{k \sqrt{K_{(i)}^2 - 1}}{2\pi T R^2} \approx \frac{J_{(i)}}{2\pi T R^2} \quad (2.38)$$

and  $L_{(i)}^j$  are the generators of the irreducible  $SU(2)$  representation of dimension  $K_{(i)}$ .

Block-diagonal matrix configurations in the matrix model (for which the equations of motion factorize into those for the individual blocks) are interpreted as describing collections of classically independent objects. In the present case, (2.36)-(2.37) represent distinct fuzzy spheres. More precisely the block diagonal matrices (2.36) describe a collection of concentric fuzzy spheres of radii  $r_{(i)}$  given in (2.38). They extend in the  $AdS_4$  directions and carry momentum  $J_{(i)}/R$  along a great circle in  $S^7$ .

The general solution minimising the matrix model Hamiltonian is therefore characterised by a set of integers,  $J_{(i)}$ ,  $i = 1, \dots, n$ , satisfying

$$\sum_{i=1}^n J_{(i)} = J. \quad (2.39)$$

The  $J_{(i)}$ 's must be multiples of  $k$ , because the size of the  $i$ -th block in (2.36),  $K_{(i)} = J_{(i)}/k$ ,  $i = 1, \dots, n$ , is necessarily an integer. This gives further support to our proposal that the matrix size should be  $J/k$ , since the projection associated with the  $\mathbb{Z}_k$  quotient implies that the angular momentum of each membrane in a multi-membrane configuration should be a multiple of  $k$ .

In [25] it was shown that the states in the pp-wave matrix model can be organised into multiplets of the  $SU(2|4)$  supergroup and special states belonging to BPS multiplets were identified. The vacua we described in this section were shown to belong to multiplets termed doubly-atypical in [25]. These multiplets have energies which are non-perturbatively protected. Therefore the degeneracy of the vacua corresponding to different numbers of spherical membranes is not lifted in the full quantum theory.

The theory contains distinct sectors associated with the vacuum configurations (2.36)-(2.38) and the fluctuations around them. It would be interesting to study the possibility of tunnelling connecting these sectors corresponding to different perturbative vacua<sup>9</sup>. Such an effect should be understood as corresponding to the interaction of membranes. For example,

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<sup>9</sup> The fact that the energies of the ground states are non-perturbatively protected [25] suggests that the tunnelling processes may be allowed only between excited states and not between pairs of ground states.

in a two-membrane vacuum, interactions can lead to a transfer of longitudinal momentum between the two membranes. This corresponds to a transition between an initial state characterised by two angular momenta,  $J_{(1)}$  and  $J_{(2)}$ , to a final state in which the angular momenta are  $J'_{(1)}$  and  $J'_{(2)}$ , with  $J_{(1)} + J_{(2)} = J'_{(1)} + J'_{(2)} = J$ . Similarly it is possible to have tunnelling processes corresponding to the splitting or joining of membranes. For example a single membrane with angular momentum  $J$  could split into two membranes with angular momenta  $J_{(1)}$  and  $J_{(2)}$ , with  $J_{(1)} + J_{(2)} = J$ . Since the angular momenta are quantised (being integers and multiples of  $k$ ), these transitions are not allowed in perturbation theory. We expect the effect of these tunnelling processes to be negligible compared to the the leading order perturbative corrections to the spectrum which will be discussed in section 2.3.

Let us more closely examine the formula (2.38) for the radii of the minimal energy fuzzy spheres,

$$r_{(i)} = \frac{J_{(i)}}{2\pi T R^2}, \quad (2.40)$$

where  $i = 1, \dots, n$  in a vacuum with  $n$  membranes. This shows that the size of the spherical membranes grows with their angular momentum,  $J_{(i)}$ . However, for our analysis to be valid we should require that the membranes do not extend beyond the region in which the  $\text{AdS}_4 \times S^7/\mathbb{Z}_k$  background is well approximated by the pp-wave geometry. More precisely for the pp-wave approximation to be applicable we should require that the radii  $r_{(i)}$  satisfy  $r_{(i)} \ll R$ . Using (2.3) this amounts to  $J_{(i)} \ll (Nk)^{1/2}$ . Combining this result with the requirement (2.13) that the  $J_{(i)}$ 's be large we obtain the condition

$$1 \ll J_{(i)} \ll (Nk)^{1/2} \quad (2.41)$$

for the pp-wave approximation to be valid. In section 2.3 we will discuss how a stricter lower bound on  $J$  arises if one requires that quantum corrections in the matrix model be small.

The pp-wave approximation we have discussed so far can be considered as keeping the leading order terms in an expansion in powers of

$$\frac{r}{R} \sim \left( \frac{J^2}{Nk} \right)^{\frac{1}{2}}. \quad (2.42)$$

It should be possible to compute corrections to the pp-wave approximation and incorporate higher orders in this expansion into the matrix model.

As observed above, the various perturbative vacua are expected to be non-perturbatively connected through tunnelling processes. Therefore it may be more natural to require that the pp-wave approximation be applicable to all possible vacua and not just to a particular one corresponding to a given set of  $J_{(i)}$ 's. If we take this point of view, considering the perturbative vacuum consisting of a single membrane, it follows that the total  $J$  should satisfy

$$J \ll (Nk)^{1/2}. \quad (2.43)$$

This condition in turn implies a bound on the number,  $n$ , of membranes. Since the individual  $J_{(i)}$ 's are integers and multiples of  $k$ , the vacuum with the largest number of membranes with a given total  $J$  corresponds to the case in which  $J_{(i)} = k$  for all  $i = 1, \dots, n$ . Combining (2.43) and (2.39) for this vacuum we get

$$J = \sum_{i=1}^n J_{(i)} = nk \ll (Nk)^{1/2}, \quad (2.44)$$

and thus

$$n \ll (N/k)^{1/2}. \quad (2.45)$$

This condition is consistent with the fact that we are describing configurations of membranes in a fixed background, obtained as near-horizon limit of a black brane solution corresponding to  $N$  coincident membranes, without including any back-reaction.

At first sight, requiring the validity of the pp-wave approximation for all possible perturbative vacua may appear to be incompatible with the lower bound in (2.41). Considering for simplicity  $k = 1$ , in the extreme case in which  $J_{(i)} = 1$  for all  $i$ , the condition  $J_{(i)} \gg 1$  is not satisfied, implying that the vacuum fluctuations of the centre of mass motion of the membranes will invalidate the use of the pp-wave approximation, as explained in the general discussion around (2.13). However, this problem may be resolved if we use a dual description of this membrane configuration in terms of M5-branes, using the proposal in [23, 48]. According to these papers the vacuum corresponding to the partition  $J = 1 + \dots + 1$ , should be identified with a configuration of a single M5-brane. Since the angular momentum of the M5-brane is  $J$ , (2.13) is satisfied from the M5-brane point of view. The size of this M5-brane is given by [48]

$$r_{M5}^4 \sim \frac{J}{R^2} \ell_P^6. \quad (2.46)$$

Similarly to the condition (2.41), the validity of the pp-wave approximation for the M5-brane requires  $r_{M5}/R \ll 1$ . Using (2.3), this amounts to

$$J \ll Nk, \quad (2.47)$$

which is satisfied automatically in our regime.

Similar considerations can be applied to other states containing multiple membranes with small angular momentum, which can be identified with configurations of M5-branes carrying large angular momentum, satisfying the conditions of applicability of the pp-wave approximation. For intermediate values of the angular momenta, more complicated configurations, such as the five-branes discussed in [49], may be relevant.

## 2.2 Near BPS fluctuations

We next consider the fluctuations around the ground states discussed in the previous section. The spectrum of such fluctuations for the pp-wave matrix model has been studied in detail in [24, 51]. We will present the results in terms of parameters,  $R$ ,  $J$  and  $k$ , which are suitable for the comparison with the ABJM theory to be discussed in section 3.

We focus on the single membrane vacuum, *i.e.* the case in which the minimal energy configuration corresponds to matrices  $Y_0^i$  of the form (2.32), where the  $L^i$ 's are the generators of the  $K = J/k$  dimensional irreducible representation of  $SU(2)$ . The case of multi-membrane vacua will be discussed in section 4.

In order to study the spectrum of excited states in the single membrane sector we expand all the fields in terms of fluctuations about the classical solution ( $X_0^m = 0, Y_0^i, \Psi_0 = 0$ ). For the  $Y^i$  scalars, which are the only variables with a non-trivial background value, we denote the fluctuation by  $Y'^i$  and write

$$Y^i = Y_0^i + Y'^i. \quad (2.48)$$

Substituting into the matrix model Hamiltonian we get quadratic, cubic and quartic terms in the fluctuations,

$$H = H^{(2)} + H^{(3)} + H^{(4)}. \quad (2.49)$$

The tree-level spectrum is determined by computing the eigensystem of the quadratic Hamiltonian,  $H^{(2)}$ , which takes the form

$$H^{(2)} = \text{tr} \left\{ \frac{R}{2k} (P_\alpha)^2 + \frac{2k}{R} \left[ \left( \frac{Y'^i}{R} + i(2\pi T) \frac{R}{2k} \varepsilon^{ijk} [Y_0^j, Y'^k] \right)^2 + \frac{1}{4R^2} (X^n)^2 \right. \right. \\ \left. \left. - (2\pi T)^2 \frac{R^2}{4k^2} [X^n, Y_0^i]^2 \right] + (2\pi T) \frac{R}{2k} \Psi^T \gamma^i [Y_0^i, \Psi] - \frac{3i}{4R} \Psi^T \gamma^{123} \Psi \right\}. \quad (2.50)$$

This Hamiltonian is diagonalised by expanding the fluctuations,  $Y'^i$ ,  $X^m$  and  $\Psi$ , in a basis of  $K \times K$  matrices, which consists of discretised versions of the spherical harmonics [3, 4]. This should be expected, since the matrix model is the regularised version of the continuum membrane theory. In the continuum the vacuum solution is a spherical membrane and the spherical harmonics are the natural basis to use to expand its fluctuations. The discretised versions of the spherical harmonics are referred to as matrix spherical harmonics. They are classified by a pair of quantum numbers,  $(l, m)$ , where  $l = 0, 1, \dots, K-1$ , and  $m = -l, -l+1, \dots, 0, 1, \dots, l$ . The excited states in the matrix model spectrum are correspondingly labelled by integers  $l, m$ .

For the scalars associated with  $S^7$  directions,  $X^n$ , there are 6 polarisations and the spectrum is

$$\omega = \frac{1}{R} \sqrt{1 + 4l(l+1)} = \frac{2}{R} \left( \frac{1}{2} + l \right), \quad l = 0, 1, \dots, K-1. \quad (2.51)$$

The upper bound on the quantum number  $l$  reflects the effect of discretisation introduced by the matrix regularisation: matrix spherical harmonics constructed from the generators of the  $K$  dimensional irreducible representation of  $\text{SU}(2)$  exist only with  $l < K$ . Each level in (2.51) has a degeneracy  $(2l+1)$ , corresponding to the allowed values of the quantum number  $m$ .

We note that the mass term and the contribution from the Laplacian (1 and  $4l(l+1)$  respectively under the square root in (2.51)) combine in such a way as to result in a rational energy spectrum. The same is true for the spectrum of the  $Y'^i$  fluctuations and the fermions that we present below. This fact does not seem to have a simple explanation in the matrix model. However, we will see in section 3.2 that it has a simple interpretation on the CFT side.

The three scalars coming from  $\text{AdS}_4$  directions,  $Y'^i$ , contain only two physical transverse polarisations. This is because of the presence of the constraint (2.30) associated with the residual gauge symmetry corresponding to area preserving diffeomorphisms<sup>10</sup>. Diagonalising the quadratic Hamiltonian in this sector yields energies

$$\omega = \frac{2}{R} (1 + l), \quad l = 0, 1, \dots, K-2 \quad (2.52)$$

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<sup>10</sup>Since the membranes are point-like in the  $S^7$  directions, fluctuations of all  $X^m$ 's,  $m = 4, \dots, 9$ , are transverse and thus there are six associated polarisations.

and

$$\omega = \frac{2}{R} l, \quad l = 1, 2, \dots, K \quad (2.53)$$

respectively for the two sets of states. For each of the energies (2.52) and (2.53) the degeneracy of the corresponding states is  $(2l + 1)$ .

In order to study the fermionic fluctuations one first decomposes the  $\text{SO}(9)$  Majorana spinors according to the  $\text{SO}(3) \times \text{SO}(6)$  isometries of the pp-wave matrix model. Diagonalising the quadratic Hamiltonian yields two sets of states with energies respectively

$$\omega = \frac{2}{R} \left( \frac{3}{4} + j \right), \quad j = \frac{1}{2}, \frac{3}{2}, \dots, K - \frac{3}{2} \quad (2.54)$$

and

$$\omega = \frac{2}{R} \left( \frac{1}{4} + j \right), \quad j = \frac{1}{2}, \frac{3}{2}, \dots, K - \frac{1}{2}. \quad (2.55)$$

The multiplicity of the corresponding states is  $4 \times (2j + 1)$  for both sets, with the factor of 4 coming from the fact that the fermions are spinors of the  $\text{SO}(6)$  isometry group associated with rotations in the transverse  $S^7$  directions.

The spectrum of the pp-wave matrix model is summarised in table 1.

Type	Label	Energy ( $\omega$ )	Multiplicity
$S^7$ scalars, $X^n$	$l = 0, 1, \dots, K - 1$	$\frac{2}{R} \left( \frac{1}{2} + l \right)$	$6 \times (2l + 1)$
AdS <sub>4</sub> scalars, $Y'^i$	$l = 0, 1, \dots, K - 2$	$\frac{2}{R}(1 + l)$	$(2l + 1)$
	$l = 1, 2, \dots, K$	$\frac{2}{R} l$	$(2l + 1)$
Fermions, $\Psi$	$j = \frac{1}{2}, \frac{3}{2}, \dots, K - \frac{3}{2}$	$\frac{2}{R} \left( \frac{3}{4} + j \right)$	$4 \times (2j + 1)$
	$j = \frac{1}{2}, \frac{3}{2}, \dots, K - \frac{1}{2}$	$\frac{2}{R} \left( \frac{1}{4} + j \right)$	$4 \times (2j + 1)$

Table 1: Spectrum of pp-wave matrix model near-BPS excitations

In section 3 we will compare these results with the energies of the dual states in the radially quantised ABJM theory. The comparison is done using the relation

$$\omega = \frac{\Delta}{R'} - \frac{J_4}{R} = \frac{1}{R}(2\Delta - J_4) \quad (2.56)$$

between the matrix model energies,  $\omega$ , and the parameters  $\Delta$  and  $J$  characterising the CFT operators.

### 2.3 Perturbation theory

Quantum corrections to the energy spectrum reviewed in the previous subsection are computed using standard quantum mechanics perturbation theory. The majority of the fluctuations are non-BPS and therefore their spectrum will be corrected, but there are some

BPS fluctuations whose spectrum is protected [25]. The situation is reminiscent of the open string spectrum around giant gravitons in the pp-wave approximation [50]. Leading order corrections for some of the states in the spectrum were computed in [24, 51].

The perturbation part of the Hamiltonian consists of cubic and quartic terms in the fluctuations around the classical solution. Expanding the Hamiltonian (2.27) we get

$$H^{(3)} = (2\pi T) \frac{R}{2k} \text{tr} \left\{ 2[X^m, Y_0^i][X^m, Y'^i] + 2[Y_0^i, Y'^j][Y'^i, Y'^j] + i \frac{2k}{R^2} \varepsilon_{ijk} Y'^i [Y'^j, Y'^k] \right. \\ \left. + \Psi^T \gamma^m [X^m, \Psi] + \Psi^T \gamma^i [Y'^i, \Psi] \right\} \quad (2.57)$$

and

$$H^{(4)} = -(2\pi T) \frac{R}{4k} \text{tr} \left\{ [X^m, X^n]^2 + 2[X^m, Y'^i]^2 + [Y'^i, Y'^j]^2 \right\}. \quad (2.58)$$

The leading order correction to the energy of a generic state,  $|n\rangle$ , is computed using the familiar formula

$$\Delta E_n = \sum_{n'} \frac{\langle n | H^{(3)} | n' \rangle \langle n' | H^{(3)} | n \rangle}{E_n - E_{n'}} + \langle n | H^{(4)} | n \rangle. \quad (2.59)$$

Note in particular that the degeneracy of the un-perturbed states due to the  $\text{SO}(3)$  symmetry does not require the use of degenerate perturbation theory as the perturbed Hamiltonian still possesses the  $\text{SO}(3)$  symmetry. We will not present explicit perturbative calculations in the pp-wave matrix model. We will limit ourselves to recalling the relative weight of the perturbative corrections compared to the tree level result. This was studied in [24, 51] and we present here the result in terms of parameters which are more suitable in the AdS/CFT context for the comparison with the ABJM theory. The tree level energies summarised in table 1 are of order  $1/R$ . The ratio of the one loop corrections (2.59) to the tree level result is of order [24, 25, 51]

$$\frac{T^2 R^6}{J^3} \sim \frac{Nk}{J^3}, \quad (2.60)$$

where we used (2.2) and (2.3) and omitted numerical factors. Hence, quantum corrections in the pp-wave matrix model are small when

$$J \gg (Nk)^{1/3}. \quad (2.61)$$

The first term in (2.59) involves a sum over intermediate states and both terms contain sums over the  $l$  and  $m$  quantum numbers arising from the expansion in matrix spherical harmonics. Each summand is the matrix element between individual states. The matrix elements, estimated using the Hamiltonian (2.26), are of order  $Nk/J^3$ , which is the same as (2.60). The fact that the matrix elements are small for large  $J$  is expected since the strong centrifugal force for large  $J$  suppresses the fluctuations (see (2.11)) making the interaction terms smaller than the quadratic terms. However, the sums in (2.59) can potentially produce factors of  $K = J/k$  and alter (2.60). Hence the dependence of the loop corrections on  $J$  is the result of the competition of two effects: as  $J$  grows, each matrix element is suppressed, but at the same time the number of degrees of freedom increases. The explicit calculations in [24, 51] show that at leading order no extra factors of  $K$  arise from the summations, thanks to non-trivial cancellations due to supersymmetry. This was proven in [25] for all states in the single membrane vacuum and it is natural to expect (2.60) to hold for the leading order

corrections in all vacua. The absence of extra factors of  $K$  in the perturbative expansion at leading order is related to the one loop finiteness of the membrane world-volume theory in the matrix regularisation, where the size of the matrices,  $K$ , plays the role of UV cut-off. Further work is needed to establish whether similar cancellations persist at higher orders, so that (2.60) can be considered a genuine coupling constant for the pp-wave matrix model.

The ratio (2.60) can also be related to the ratio of the eleven dimensional Planck length to the size of the spherical membranes,

$$\frac{Nk}{J^3} \sim \left( \frac{l_P}{r} \right)^3, \quad (2.62)$$

where  $r$  is given in (2.40). This is natural, since in a theory of quantum gravity, such as M-theory, loop corrections should be suppressed when the relevant length scale is much larger than the Planck scale. Only when  $J$  is sufficiently large such that  $Nk/J^3 \ll 1$ , it is possible to distinguish the extended spherical membranes we are discussing from point-like gravitons.

We can also estimate the size of the fluctuations of the membrane coordinates around the stable fuzzy sphere. These fluctuations should be small compared to the radius of the sphere. The magnitude of the fluctuations of the membrane coordinates is of order  $R/\sqrt{J}$ , as can be deduced from the simple particle picture of the pp-wave approximation presented at the beginning of section 2. The ratio of this to the size of the spherical membrane,  $J/TR^2$ , is therefore  $R^3/J^{3/2} = (Nk/J^3)^{1/2}$ . Hence one gets the same condition,  $J \gg (Nk)^{1/3}$ , that we deduced from the suppression of loop corrections.

Combining the condition of applicability of the pp-wave approximation (2.41) with the requirement that quantum corrections be small leads to the condition

$$(Nk)^{1/3} \ll J \ll (Nk)^{1/2}. \quad (2.63)$$

Therefore the sector of M-theory states in  $\text{AdS}_4 \times S^7/\mathbb{Z}_k$  characterised by angular momentum  $J$  satisfying (2.63) can be reliably studied perturbatively using the pp-wave matrix model. In the next section we will argue that in this regime a suitable perturbative expansion scheme can be developed for the ABJM theory as well.

In section 2.1 we noticed that the use of the pp-wave approximation for all possible perturbative vacua can be justified if a dual description of certain vacua in terms of M5-branes following [48] is employed. The quantum fluctuations of the M5-brane should be suppressed when

$$r_{M5} \gg l_P. \quad (2.64)$$

Recalling the formula (2.46) for the radius of the M5-brane, we find that (2.64) is satisfied when the condition  $J \gg (Nk)^{1/3}$  found above holds. This does not contradict the fact that the  $J = 1 + \dots + 1$  vacuum cannot be treated perturbatively, since we lack a direct classical description of the degrees of freedom of M5-branes in the matrix model.

We note the formula

$$\frac{Nk}{J^3} \frac{J^2}{Nk} = \frac{1}{J} \quad (2.65)$$

which suggests that  $1/J$  corrections to the computation we have described may be understood in terms of a double expansion in powers of  $Nk/J^3$ , which controls the loop corrections, and of  $J^2/Nk$ , which controls the corrections to the pp-wave approximation. A simple class of  $1/J$  corrections arises from the distinction between the  $J_M$  and  $J_4$  generators. The parameter that



we denoted by  $J$  in the continuum membrane Hamiltonian is the eigenvalue of  $J_4$ , whereas in the matrix regularisation we used matrices of size  $J/k$ , related to the eigenvalue of  $J_M$ . On the states discussed in section 2.2  $J_M$  and  $J_4$  differ by an amount of order 1, which can be neglected in our analysis. Keeping track of this difference would result in  $1/J$  corrections to the spectrum.

Our prescription of using  $K \times K$  matrices in the regularisation of the large angular momentum sector we focus on implies that the number of degrees of freedom in the resulting quantum mechanical system is of order  $K^2 \sim (J/k)^2$ . In the next section we will show that in the dual sector of the ABJM theory a number of states of order  $K^2$  arises naturally within the Born-Oppenheimer scheme. This matching of the number of degrees of freedom between the two sides of the correspondence lends additional support to our proposal. However, this observation should be taken cautiously because the matrix model can be expected to provide a good approximation to the continuum theory only for low-lying states with quantum number  $l \ll K$ . For  $l$  approaching  $K$  one expects the discretised description of membranes in terms of matrices to provide a poor approximation. To be more precise, in order for the approximation (2.22) of the Lie bracket  $\{f_1, f_2\}$  by the matrix commutator to be valid, we need the condition  $k_1 k_2 \ll K$ , where  $k_1$  and  $k_2$  refer to the wave numbers (in the sphere case, the label of the spherical harmonics  $l$ ) characterising the scale of variation of the functions  $f_1$  and  $f_2$ <sup>11</sup>. This at first sight seems to put a rather stringent condition  $l \ll \sqrt{K}$ . However, the leading order terms contributing to the dynamics of the fluctuations around the stable solution are given by the Lie brackets between the background solution itself (which has wave number of order 1) and the fluctuation, with wave number  $l$ . This leads to the condition  $l \ll K$ . A similar restriction on the angular momentum quantum numbers also appears in the analysis of the CFT side, where the effective description we use is good only for states with energies much smaller than that of the high energy states that we will argue should be integrated out.

### 3 CFT side

In this section we will study the CFT side of the correspondence in the large  $J$  regime.

Let us first recall some general features of the theory which was proposed in [13] as CFT dual to M-theory in  $\text{AdS}_4 \times S^7/\mathbb{Z}_k$ . The ABJM theory is a  $U(N) \times U(N)$  Chern-Simons gauge theory with matter in the bi-fundamental representation. The field content consists of four bi-fundamental complex scalar fields,  $(\phi^A)^i_{\hat{j}}$ , four bi-fundamental complex spinors,  $(\psi^A)^{\hat{i}}_j$ ,  $A = 1, \dots, 4$ , in addition to the gauge fields,  $(A_\mu)^i_j$  and  $(\hat{A}_\mu)^{\hat{i}}_{\hat{j}}$ , associated with the two  $U(N)$  factors. Here  $i = 1, \dots, N$  and  $\hat{i} = 1, \dots, N$  are colour indices referring to the two  $U(N)$ 's. The conformal dimension of fermions and gauge fields is 1, while the scalars have dimension  $1/2$ . The two Chern-Simons terms in the theory have (integer) level  $k$  and  $-k$  respectively and  $1/k$  plays the role of expansion parameter in a standard perturbative treatment.

For generic  $k$  the theory has a  $SU(4) \times U(1)$  R-symmetry corresponding to the isometry group of  $S^7/\mathbb{Z}_k$  in the dual M-theory background. The  $U(1)$  factor is generated by  $J_M$  corresponding to translations along the M-theory circle and the  $SU(4)$  factor is the remaining part of the  $SO(8)$  isometries of  $S^7$ , which commute with  $J_M$ . Because of the  $\mathbb{Z}_k$  quotient

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<sup>11</sup>We note that in the torus case [52–54], the approximate equality between the Lie brackets and the matrix commutator follows from the condition  $\sin(\frac{\pi}{K} \vec{k}_1 \times \vec{k}_2) \approx \frac{\pi}{K} \vec{k}_1 \times \vec{k}_2$ , where  $\vec{k}_1$  and  $\vec{k}_2$  denote the two dimensional wave number vectors on the torus.

the full  $\text{SO}(8)$  symmetry is present only for  $k = 1, 2$ . We will use a component formulation of the ABJM theory similar to that given in [55], in which the  $\text{SU}(4)$  symmetry and the corresponding  $\mathcal{N} = 6$  supersymmetry are manifest<sup>12</sup>. The full  $\mathcal{N} = 8$  supersymmetry is believed to be recovered in the special case of  $k = 1, 2$ .

The matter fields  $\phi^{Ai}_{\hat{j}}$ ,  $\psi^{Ai}_j$  transform in the fundamental of  $\text{SU}(4)$ . They transform under the symmetry generated by  $J_M$  as

$$\left(\phi^{Ai}_{\hat{j}}\right)' = e^{i\alpha}\phi^{Ai}_{\hat{j}} \quad (3.1)$$

$$\left(\psi^{Ai}_j\right)' = e^{-i\alpha}\psi^{Ai}_j \quad (3.2)$$

where  $\alpha$  is the parameter of the transformation [13]<sup>13</sup>. The symmetry generators  $J_1, J_2, J_3$  and  $J_4$  which we introduced in section 2 (corresponding to rotations in the 12, 34, 56 and 78 planes of  $\mathbb{R}^8$  in which the  $S^7$  is embedded) are realised in the ABJM theory as certain linear combinations of  $J_M = J_1 + J_2 + J_3 + J_4$  and the Cartan generators of  $\text{SU}(4)$ . Even for  $k \neq 1, 2$  the currents associated with the generators  $J_1, J_2, J_3$  and  $J_4$  are still conserved, although the full  $\text{SO}(8)$  symmetry is broken to  $\text{U}(1) \times \text{SU}(4)$ . It is important for our construction that the scalar field  $\phi^4$  has unit charge under  $J_4$  and is not charged with respect to  $J_1, J_2$  and  $J_3$ .

In section 2, we studied a sector of M-theory in  $\text{AdS}_4 \times S^7/\mathbb{Z}_k$  consisting of states for which  $J_4$  is large and  $J_1, J_2, J_3$  are of order 1. We now wish to construct the corresponding gauge-invariant operators on the CFT side. They are characterised by having large R-charge  $J_4$ . This can be achieved by considering operators involving a large number of  $\phi^4$  insertions, since this field has unit charge under  $J_4$ . By construction this results in a large  $J_M = J_1 + J_2 + J_3 + J_4$  as well. The definition of gauge invariant composite operators with non-zero  $J_M$  charge in the ABJM theory involves the use of so-called monopole operators [13, 27, 28], closely related to the disorder operators introduced in [26]. Monopole operators play an important role in the ABJM theory and more generally in three dimensional gauge theories [13, 27, 28]. They are crucial for example for the enhancement of supersymmetry to  $\mathcal{N} = 8$  for  $k = 1, 2$  in the ABJM theory [56–58].

Monopole operators have no simple realisation as local polynomials in the elementary fields and the most convenient way of describing them in a conformal field theory is using radial quantisation and the state-operator map [27, 28]. In the case of a three-dimensional CFT this involves mapping local operators inserted at the origin to states in a Hamiltonian formulation with the radial direction interpreted as Euclidean time. The Hamiltonian in radial quantisation, for which each time slice is an  $S^2$ , is equivalent to the dilation operator of the theory in  $\mathbb{R}^3$  and thus operators with definite scaling dimension are in one to one correspondence with eigenstates of the Hamiltonian. The requirement of gauge-invariance for the composite operators is translated into Gauss law constraints which the physical states in radial quantisation should satisfy.

In the present case the use of radial quantisation has the added benefit of leading to a more direct and natural correspondence between states in the matrix model discussed in

<sup>12</sup>This  $\text{SU}(4) \sim \text{SO}(6)$  symmetry should not be confused with the  $\text{SO}(6)$  symmetry of the pp-wave matrix model in section 2. They are embedded into the full  $\text{SO}(8)$  symmetry in inequivalent ways.

<sup>13</sup>This may be understood as the matter part of a constant gauge transformation in which the  $\text{U}(1)$  parts of the two  $\text{U}(N)$  gauge groups are assigned opposite charges. When the transformation parameter is equal to  $2\pi/k$ , the two states related by the transformation are indistinguishable [13].

section 2 – describing fluctuations of spherical membranes – and states in the ABJM theory on  $S^2$ .

The relevance of monopole operators in the ABJM theory has been observed by various authors and the use of radial quantisation has been advocated before [56, 57, 59–63]. However, the approaches proposed so far are not suitable to study the aspects of the  $\text{AdS}_4/\text{CFT}_3$  duality we are interested in. We consider small  $k$  (in order to be in a genuinely M-theoretic regime) and therefore perturbation theory in  $1/k$  is not applicable. Moreover we are especially interested in studying non-BPS states for which we cannot rely on exact non-renormalisation properties induced by supersymmetry. We will argue, however, that focussing on a large  $J$  sector makes a quantitative comparison with the matrix model possible.

In the remaining part of section 3 we will construct and study operators corresponding to the states on the AdS side discussed in section 2, using monopole operators and the state-operator map. Since parts of the following discussion will be rather technical, we first present a brief summary of our analysis in order to highlight the essential points and emphasise the main line of ideas.

### Summary of analysis on the CFT side

We begin section 3.1 with the description of the Hamiltonian formulation of the radially quantised ABJM theory. In order to construct states corresponding to operators with large  $J_4$  charge, one has to excite the  $\phi^4$  field  $J$  times. The colour charge density associated with the  $U(N) \times U(N)$  gauge group will have expectation value of order  $J$  on the resulting state. In Chern-Simons theories coupled to matter fields, the Gauss law constraints equate the charge density to the magnetic field. Hence, in our states we should have a magnetic flux with strength of order  $J$  through the  $S^2$  space corresponding to a fixed (Euclidean) time slice in radial quantisation. The presence of this magnetic flux, which satisfies a Dirac quantisation condition [64, 65], defines the monopole operators in the framework of the state-operator map.

As we explain in section 3.1, the magnetic flux we consider is in the Abelian, diagonal, part of the two  $U(N)$  gauge groups. The total magnetic flux, which equals  $J$  (up to  $O(1)$  factors), can be distributed among the  $N$  entries of the diagonal part of the field strength (which is an  $N \times N$  matrix). The integers characterising this partition of  $J$  are referred to as the GNO charges [27–29]. The set of possible GNO charges gives the classification of the BPS operators/states on the CFT side (3.19). This characterisation of the BPS states in the ABJM theory is in direct correspondence with the classification of the ground states on the AdS side (2.39).

In section 3.2 we consider the fluctuation spectrum around a particular ground state. In order to do this, it is necessary to fix a gauge. Our gauge fixing conditions (which involve a combination of background, unitary and Coulomb gauges) are specified in (3.25)–(3.26), (3.30)–(3.32) and (3.36)–(3.38). We then compute the Hamiltonian of the gauge-fixed theory by solving the Gauss law constraints. The part of the gauge-fixed Hamiltonian necessary for the computation of the spectrum is given in (3.62). We find that the spectrum contains (i) low-energy modes with eigenfrequencies of order 1 residing in the diagonal entries of the  $N \times N$  fields, summarised in table 3, and (ii) high energy modes with eigenfrequencies of order  $J$  associated with off-diagonal entries, summarised in table 4. The spectrum of low-energy modes agrees with the spectrum we found on the AdS side, which is summarised in table 1.

We then explain in section 3.3 that this large separation of energy scales between diagonal

and off-diagonal modes suggests that a Born-Oppenheimer type approximation is applicable. In this scheme the off-diagonal, high-energy, modes should be integrated out. The calculation of the spectrum in section 3.2 is then justified as arising from the leading order in this approximation.

### 3.1 BPS states

We begin this section by presenting the radial quantisation of the ABJM theory and introducing our notation and conventions. Properties of BPS observables in the (deformed) ABJM theory have been studied by various authors using radial quantisation [56, 57, 59–62]. The derivation of the radially quantised ABJM theory in the Hamiltonian formalism requires special care with regard to the complex conjugation of fermionic fields. We have worked it out starting from the action of the ABJM theory in the component form given in [55] using slightly different conventions. In our conventions the Hamiltonian is

$$H = \int d\theta d\varphi \operatorname{Tr} \left[ \frac{1}{\sin \theta} \pi^{*A} \pi_A + \sin \theta g^{\alpha\beta} D_\alpha \phi^*{}_A D_\beta \phi^A + \frac{1}{4} \sin \theta \phi^*{}_A \phi^A + \sin \theta V_6 + \sin \theta V_Y \right. \\ \left. + \sin \theta \psi^{*T}{}_A (\sigma^r \sigma^\alpha D_\alpha - 1) \psi^A \right], \quad (3.3)$$

where the sextic scalar potential,  $V_6$ , and the Yukawa couplings,  $V_Y$ , are

$$V_6 = \left( \frac{2\pi}{k} \right)^2 \left( \frac{1}{3} \phi^B \phi^*{}_D \phi^D \phi^*{}_C \phi^C \phi^*{}_B + \frac{1}{3} \phi^B \phi^*{}_B \phi^C \phi^*{}_C \phi^D \phi^*{}_D + \frac{4}{3} \phi^B \phi^*{}_D \phi^C \phi^*{}_B \phi^D \phi^*{}_C \right. \\ \left. - 2 \phi^B \phi^*{}_B \phi^D \phi^*{}_C \phi^C \phi^*{}_D \right), \quad (3.4)$$

$$V_Y = \frac{2\pi}{k} \left( -2 \psi^{*T}{}_A \sigma^r \psi^B \phi^A \phi^*{}_B + 2 \psi^{*T}{}_A \sigma^r \phi^*{}_B \phi^A \psi^B - \psi^{*T}{}_A \sigma^r \phi^*{}_B \phi^B \psi^A + \psi^{*T}{}_A \sigma^r \psi^A \phi^B \phi^*{}_B \right. \\ \left. - i \epsilon^{ABCD} (\psi^{*T}{}_D B^{*T}) \phi^*{}_C \psi^*{}_B \phi^*{}_A + i \epsilon_{ABCD} \phi^D \psi^{CT} \phi^B (B \psi^A) \right). \quad (3.5)$$

The conventions used in the above expression are as follows. The  $\pi$ 's are the canonical conjugate variables of the  $\phi$ 's. The indices  $\alpha, \beta = 1, 2$  are used for the  $\theta, \varphi$  coordinates of the  $S^2$  time-slice in radial quantisation. In Chern-Simons theory the gauge fields  $A_1$  ( $A_\theta$ ) and  $A_2$  ( $A_\varphi$ ) are canonically conjugate to each other and the same is true for the components of  $\hat{A}_\alpha$ . We use Pauli matrices in polar coordinates,

$$\sigma^r = \begin{bmatrix} \cos \theta & \sin \theta e^{-i\varphi} \\ \sin \theta e^{i\varphi} & -\cos \theta \end{bmatrix}, \quad \sigma^\theta = \begin{bmatrix} -\sin \theta & \cos \theta e^{-i\varphi} \\ \cos \theta e^{i\varphi} & \sin \theta \end{bmatrix}, \quad \sigma^\varphi = \frac{1}{\sin \theta} \begin{bmatrix} 0 & -ie^{-i\varphi} \\ ie^{i\varphi} & 0 \end{bmatrix}. \quad (3.6)$$

In the last line of (3.5)  $B$  is the charge conjugation matrix. In the standard representation of the Pauli matrices we are using,  $B = \sigma^2$ , where  $\sigma^2$  is the usual Pauli matrix. The superscript  $T$  indicates transposition of the spinors. We use the  $*$  symbol to signify the operation in which one takes the complex conjugation (the adjoint for quantum mechanical operators) and the transpose in the colour indices. For example, we have

$$\phi^*{}_A \hat{i}_j = (\phi^{Aj} \hat{i})^*. \quad (3.7)$$

The  $*$  symbol is redundant as one can see immediately from the position of the flavour index whether the complex conjugate is implied. We will hereafter omit the  $*$  symbol when it is appropriate. Our conventions for the covariant derivative and the field strength are

$$D_\alpha \phi^A = \partial_\alpha \phi^A - i A_\alpha \phi^A + i \phi^A \hat{A}_\alpha, \quad (3.8)$$

$$D_\alpha \psi^A = \partial_\alpha \psi^A - i \hat{A}_\alpha \psi^A + i \psi^A A_\alpha, \quad (3.9)$$

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha - i[A_\alpha, A_\beta], \quad \hat{F}_{\alpha\beta} = \partial_\alpha \hat{A}_\beta - \partial_\beta \hat{A}_\alpha - i[\hat{A}_\alpha, \hat{A}_\beta]. \quad (3.10)$$

The Hamiltonian contains – in addition to the kinetic terms, the scalar potential (3.4) and the Yukawa couplings (3.5) – mass terms for scalars and fermions. These mass terms, arising from radial quantisation, reflect the conformal dimensions of the fields. For example scalar fields have mass  $1/2$  (when the radius of the  $S^2$  is normalised to 1).

The canonical commutation relations are given by

$$[\phi^{Ai}_{\hat{j}}(x'), \pi_B^{\hat{k}}{}_l(x'')] = i \delta^A{}_B \delta^i{}_l \delta^{\hat{k}}_{\hat{j}} \delta^2(x' - x''), \quad (3.11)$$

$$[\psi^{Ai}_{\hat{j}}{}^a(x'), \psi^T{}_B{}^k{}_{\hat{l}b}(x'')]_{+} = \frac{1}{\sin \theta} \delta^A{}_B \delta^i{}_l \delta^{\hat{k}}_{\hat{j}} \delta^a{}_b \delta^2(x' - x''), \quad (3.12)$$

$$[A_1{}^i{}_j(x'), A_2{}^k{}_l(x'')] = i \frac{2\pi}{k} \delta^2(x' - x'') \delta^i{}_l \delta^k{}_j, \quad (3.13)$$

$$[\hat{A}_1{}^{\hat{i}}{}_{\hat{j}}(x'), \hat{A}_2{}^{\hat{k}}{}_{\hat{l}}(x'')] = -i \frac{2\pi}{k} \delta^2(x' - x'') \delta^{\hat{i}}_{\hat{l}} \delta^{\hat{k}}_{\hat{j}}, \quad (3.14)$$

where  $\delta^2(x' - x'') = \delta(\theta' - \theta'') \delta(\varphi' - \varphi'')$  and  $a, b = 1, 2$  are spinor indices.

The Gauss law constraints are

$$\frac{k}{2\pi} F_{12} = \rho, \quad (3.15)$$

$$\frac{k}{2\pi} \hat{F}_{12} = -\hat{\rho}, \quad (3.16)$$

where  $\rho$  and  $\hat{\rho}$  are the colour charge densities for the two  $U(N)$  gauge groups,

$$\rho^i{}_j = (i \phi^A \pi_A - i \pi^A \phi_A + \sin \theta \psi^T{}_A \psi^A)^i{}_j, \quad (3.17)$$

$$\hat{\rho}^{\hat{i}}{}_{\hat{j}} = (i \phi_A \pi^A - i \pi_A \phi^A)^{\hat{i}}{}_{\hat{j}} - \sin \theta \psi^T{}_A{}^k{}_{\hat{j}} \psi^{A\hat{k}}{}_k. \quad (3.18)$$

The relative sign difference in the Gauss law constraints reflects the fact that the Chern-Simon levels for the two  $U(N)$  gauge fields are  $k$  and  $-k$ .

The idea of the state-operator map is that the eigenstates of the Hamiltonian (3.3) correspond to operators with definite scaling dimensions, which are given by the eigenvalues of the Hamiltonian. The Gauss law constraints, (3.15) and (3.16), give the condition that the states, and hence the operators, be gauge invariant.

We now discuss the construction of states corresponding to BPS operators with large charge  $J_4$ , which will involve the introduction of monopole operators. The role of monopole operators in the definition and classification of certain BPS operators in the ABJM theory was first discussed in [13]. For further discussion, see [56, 57, 59–63]. The structure of BPS operators studied in these papers is consistent with the dual AdS picture. The BPS operators we discuss below are a special case, characterised by a large  $J_4$  charge, of the operators

considered in these papers. The main novelty in our work will be in extending the construction to non-BPS operators. This will be presented in the next subsection after a suitable gauge fixing.

The arguments in the remainder of this subsection leading to the identification of BPS states should be considered as heuristic. They involve assumptions, which may be difficult to directly justify. In section 3.2, however, we will give an alternative description of these states. This is achieved by a suitable choice of gauge in which each BPS state becomes the simple perturbative vacuum. The gauge choice is motivated by the description of BPS states in this subsection. The analysis in section 3.2 of these BPS ground states – and also of the non-BPS excited states around them – is reliable in the large  $J$  sector, provided that the approximation scheme we propose, which will be explained in section 3.3, is valid.

The BPS operators we are interested in have minimum conformal dimensions for given R-charges. Therefore, using the state-operator map, we will look for states which have minimum energies, *i.e.* the ground states in the sector with given  $J_4$ . Our first assumption is that these ground states can be identified using a free field description in which we treat the theory as if it consisted of a collection of harmonic oscillators, neglecting the interactions among them. One basis for this assumption is that we are considering BPS operators, for which observable quantities such as conformal dimensions are protected. Under this assumption, states for which the R-charge  $J_4$  takes a given value  $J$  can be obtained by exciting the field  $\phi^4$   $J$  times.

We should excite only zero modes of  $\phi^4$  on the  $S^2$  time-slice in radial quantisation, since non-zero modes have larger energy and hence their excitation should be avoided. The state thus obtained has constant colour charge densities,  $\rho$  and  $\hat{\rho}$ , of order  $J$ , because the  $\phi^4$  field carries non-zero colour charge<sup>14</sup>. Hence from the Gauss law constraints, (3.15) and (3.16), one has to introduce a constant magnetic flux through the  $S^2$  time-slice in radial quantisation. This magnetic flux defines the monopole operator.

Since the  $\phi^4$  field is a matrix,  $\phi^{4i}_{\hat{j}}$ , we should specify which elements of the matrix contribute to the ground state. Our second assumption is that only diagonal elements of  $\phi^{4i}_{\hat{j}}$  should be excited. The need of a similar assumption was pointed out in [59]. The states for which off-diagonal elements are excited should either have larger energy or be gauge-equivalent to the states obtained by diagonal excitations only. The rationale for this assumption is the fact that the non-negative scalar potential (3.4) is classically zero when the fields consist of diagonal matrices (or configurations gauge-equivalent to diagonal matrices). Moreover, classically any configuration of the matrix field  $\phi^{4i}_{\hat{j}}$  can be diagonalised by a  $U(N) \times U(N)$  gauge transformation, as can be proven using the so-called polar decomposition of matrices [66].

By the above assumptions, it is sufficient to excite only zero modes in the diagonal entries of  $\phi^4$  and we denote by  $J_{(i)}$  the number of excitations of the  $i$ -th diagonal component. Since the total number of excitations of  $\phi^4$  should be  $J$ , the possible BPS states are labelled by a set of integers satisfying

$$J = \sum_{i=1}^N J_{(i)}, \quad (3.19)$$

where each  $J_{(i)}$  turns out to be a multiple of  $k$  as we will see below. As pointed out above, this excitation of zero modes induces a constant flux, through the Gauss law constraints (3.15)-

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<sup>14</sup>This state may be interpreted as the Bose-Einstein condensate, resulting from the requirement that the charge  $J_4$  be large.



(3.16). In order to satisfy these equations we need gauge fields  $A_\alpha$  and  $\hat{A}_\alpha$  with diagonal components given by the vector potential of a Dirac monopole [67] with magnetic charge  $J_{(i)}/2k$ ,

$$A_\alpha = \text{diag} \left[ \frac{J_{(i)}}{2k} \right] \times (A_\alpha \text{ for Dirac monopole with unit magnetic charge}) \quad (3.20)$$

$$\hat{A}_\alpha = \text{diag} \left[ \frac{J_{(i)}}{2k} \right] \times (A_\alpha \text{ for Dirac monopole with unit magnetic charge}). \quad (3.21)$$

Because of the Dirac quantisation condition, the gauge fields (3.20) and (3.21) are consistent only if all  $J_{(i)}/2k$  are integers divided by two, *i.e.* only if all the  $J_{(i)}$ 's are multiples of  $k$ . We define

$$2kq_{(i)} = J_{(i)} \quad (3.22)$$

where  $2q_{(i)}$  is an integer<sup>15</sup>. These  $q_{(i)}$ 's, which characterise the configuration of flux in radial quantisation (or equivalently the monopole operator), are referred to as GNO charges [27–29].

This classification of BPS states in terms of a set of integers,  $J_{(i)}$ , satisfying (3.19) is in direct correspondence with the classification of vacua in the pp-wave matrix model discussed in section 2.1. In the matrix model the integers  $J_{(i)}$  characterise the angular momenta of concentric spherical membranes and satisfy the condition (2.39), which is the same as (3.19).

The energy  $\Delta$  of the state resulting from the excitations of  $\phi^4$  described above is

$$\Delta = \frac{J}{2}, \quad (3.23)$$

since the only contribution to the energy comes from the mass of the field  $\phi^4$ , which is  $1/2$ . We note that in (3.3) there is no direct contribution to the energy from the magnetic field. This is as expected in view of the BPS nature of the state and it also agrees with the property of the dual ground state in the matrix model. For a more generic class of BPS operators relations analogous to (3.23) were verified in [56, 57] using a method based on an appropriate deformation of the ABJM theory.

A difference between the two sides of the duality, at least at first sight, is that in the CFT the rank of the gauge groups,  $N$ , sets an upper bound on the number of non-zero integers in the partition of  $J$ : the number of GNO charges cannot exceed  $N$ , which in particular implies  $J \leq N$ . Such a bound need not be satisfied on the gravity side. For example, in a vacuum with a very large number  $n$  of membranes, for which the individual angular momenta  $J_{(i)}$  obey (2.41), the sum  $\sum_{i=1}^n J_{(i)} = J$  can be larger than  $N$ . This is not an inconsistency, as can be understood using the following observation. The states on the AdS side violating the upper bound would contain more than  $N$  spherical membranes with angular momentum in  $S^7$ . However, the original  $\text{AdS}_4 \times S^7$  background is produced by a stack of  $N$  membranes and it is natural to expect that neglecting the back-reaction of a number of rotating membranes larger than  $N$  on this background would be inconsistent. Therefore, the study of configurations of this type would be outside of the validity of the usual AdS/CFT correspondence. In fact, this is a general feature common to all examples of AdS/CFT duality. In a gauge theory for finite

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<sup>15</sup> We follow the convention in [67] for the definition of the magnetic flux  $q$ , where  $2q$  is an integer. The spectrum of the Laplace operator on  $S^2$  in the presence of the magnetic flux is given by  $l(l+1) - q^2$  in this convention. In the recent literature, *e.g.* in [57, 59],  $q' = 2q$  is usually denoted by  $q$ .



$N$  there is an upper bound on the number of independent gauge-invariant combinations of fields, which in general should be understood on the AdS side as being related to the effect of the back-reaction on the background.

A more concrete resolution of the apparent contradiction can be given recalling the considerations of section 2.1 on the applicability of the pp-wave approximation. As discussed in 2.1, the possibility of tunnelling between different perturbative sectors leads to a compelling argument for requiring the applicability of the pp-wave approximation to all possible matrix model vacua. This in turn leads to the condition  $J \ll (Nk)^{1/2} < N$ , so that the upper bound on the CFT side ceases to be meaningful.

The correspondence between 1/2 BPS operators of the ABJM theory and ground states of the pp-wave matrix model was also pointed out in [61]. However, the importance of focussing on a large  $J$  sector, which is the essential ingredient that allows us to extend the analysis beyond the BPS sector, was not noticed in [61].

On the AdS side, as discussed in section 2.1, the vacuum states classified by the partition of  $J$  should be connected non-perturbatively by tunnelling effects, which physically should be interpreted as interaction of membranes. Therefore we expect the ground states considered in this section to be connected non-perturbatively as well, provided that the sum of the GNO charges,  $\sum_{(i)} q_{(i)}$ , is conserved. This presumably means that there should be classical solutions in the Euclidean theory connecting two given Dirac monopoles (with the same total GNO charge), corresponding to a tunnelling between the two configurations. It would be interesting to identify and explicitly construct such instanton-like solutions interpolating between vacua corresponding to different sets of GNO charges.

The construction in this paper is analogous to the construction of BPS operators in the BMN sector of  $\mathcal{N} = 4$  super Yang-Mills. The scalar field  $\phi^4$  plays a role similar to the complex scalar  $Z$  in [23]. However, one cannot define simple gauge invariant BMN-like operators by taking a trace because  $\phi^4$  is a bi-fundamental field. This leads to the necessity of using monopole operators in the way discussed here. This is related to a crucial difference between the Chern-Simons and Yang-Mills theories, *i.e.* the fact that in the former the Gauss law constraints (3.15)-(3.16) equate the charge density to the magnetic part of field strength, rather than the divergence of the electric field. As a consequence one can have a non-zero (although quantised by Dirac's condition) total colour charge even on a compact space (in the present case the  $S^2$  in radial quantisation).

### 3.2 Near BPS excitations

In this section, we study the near BPS fluctuations around the BPS states described in the previous section.

Before going into the details of the gauge fixing procedure we use, we discuss some basic ideas behind it. In the state-operator map, the gauge invariance of an operator translates into the Gauss law constraint imposed on the corresponding state. A clean approach to study these physical states is to first quantise the Hamiltonian formalism described in section 3.1 and then later impose the first class constraints (the Gauss law constraints) on state vectors, following Dirac's approach to the quantisation of constrained systems [68]. We found that, for the ABJM theory in the sector we are considering, it is not easy to carry out this program, due to technical difficulties associated with ordering ambiguities in the quantum Hamiltonian and the constraints. In the following, we employ another standard approach (used for example in

the quantisation of string theory in light-cone gauge [69]) to study physical states in a theory with gauge symmetries. We first eliminate some of the degrees of freedom by introducing gauge fixing conditions. We then express the conjugate momenta of the eliminated variables in terms of the remaining physical variables using the Gauss law constraints. These two steps are carried out in the classical theory, and the resulting gauge-fixed theory is then quantised. The information of the Gauss law constraints is already taken into account at the classical level and the states in the quantum theory are by construction physical. (If there is a residual gauge symmetry, the constraints cannot be solved completely. The remaining part of the constraints should be imposed on the states of the (partially) gauge-fixed theory.) The ordering problem does not arise in this approach. The price one has to pay is that there is no guarantee that the gauge-fixed theory will have all the (global) symmetries of the original theory. Although we believe that all global symmetries of the original ABJM theory are properly realised in the gauge-fixed theory we describe below, it is important to explicitly verify this.

The gauge we choose is a combination of the background, unitary and Coulomb gauges. This choice is particularly well-suited to clarify the physical content of the theory in the sector we consider. We focus on the case where only one of the GNO charges is non-zero,  $J = J_{(1)}$ , corresponding to the case where there is only one spherical membrane. We discuss some aspects of the more general case in section 4. In the presence of a single GNO charge, it is convenient to introduce indices  $i', j' = 2, \dots, N$  and  $\hat{i}', \hat{j}' = 2, \dots, N$ . Elementary  $N \times N$  fields in the ABJM theory can be decomposed into blocks of size  $1 \times 1$ ,  $(N-1) \times (N-1)$ ,  $1 \times (N-1)$  and  $(N-1) \times 1$  respectively. For example, the field  $\phi^{Ai}_{\hat{j}}$  is decomposed as

$$\phi^{Ai}_{\hat{j}} = \left[ \begin{array}{c|c} \phi^{A1}_{\hat{1}} & \phi^{A1}_{\hat{j}'} \\ \hline \phi^{Ai'}_{\hat{1}} & \phi^{Ai'}_{\hat{j}'} \end{array} \right]. \quad (3.24)$$

Since we wish to treat fluctuations around the ground state as perturbations, it is necessary to separate the large, order  $J$ , contribution of the ground state from the small, order one, contribution of the fluctuations. We accordingly split the gauge field into a background part,  $B_\alpha$ , corresponding to the constant monopole flux as explained around (3.20) and (3.21), and the fluctuation about it,  $a_\alpha$ ,

$$(A_\alpha)^i_j = (B_\alpha)^i_j + (a_\alpha)^i_j, \quad (3.25)$$

as is done in the usual background gauge. Similarly we decompose  $\hat{A}_\alpha$  as

$$(\hat{A}_\alpha)^{\hat{i}}_{\hat{j}} = (B_\alpha)^{\hat{i}}_{\hat{j}} + (\hat{a}_\alpha)^{\hat{i}}_{\hat{j}}. \quad (3.26)$$

Here the background field  $B_\alpha$  about which both  $A_\alpha$  and  $\hat{A}_\alpha$  are expanded, is

$$(B_2)^i_j = (B_2)^{\hat{i}}_{\hat{j}} = \left[ \begin{array}{c|cccc} q(1 - \cos \theta) & 0 & 0 & \cdots & 0 \\ \hline 0 & & & & \\ 0 & & & & \\ \vdots & & & & \\ 0 & & & & \end{array} \right], \quad (B_1)^i_j = (B_1)^{\hat{i}}_{\hat{j}} = 0. \quad (3.27)$$

In [67] the background gauge field is given for two patches excluding either the north-pole or the south-pole of  $S^2$ . We will use the patch excluding the south-pole. The observable quantities we compute in the following should not depend on the choice of patch.

We require that the fluctuation fields  $a_\alpha$  and  $\hat{a}_\alpha$  do not have singularities present in the gauge field for the Dirac monopole, as small fluctuations cannot satisfy the Dirac quantisation condition. Therefore formulae (3.25), (3.26) and (3.27) imply that we focus on a particular sector of the ABJM theory. More precisely, we have focussed on a perturbative vacuum associated with a particular choice of GNO charges, although as remarked in section 3.1, two given vacua with the same total GNO charge should be non-perturbatively connected by tunnelling effects.

The Gauss law constraints, (3.15) and (3.16), equate the magnetic field to the charge density. Therefore we should separate the contribution of the ground state and the fluctuation to the charge density as well. The charge density for our ground state (the state with lowest energy for given charge) is produced by the excitation of the zero-mode oscillator associated with the field  $\phi^{41}_{\hat{1}}$ , as explained in 3.1. However, the separation cannot be achieved in a straightforward manner by introducing a background value for  $\phi^{41}_{\hat{1}}$ ; the vacuum expectation value of  $\phi^{41}_{\hat{1}}$  is zero for the ground state we are considering, although the vacuum expectation values of the composite operators  $\rho$  and  $\hat{\rho}$  are non-zero. This is a consequence of the fact that the phase of  $\phi^4$  is undetermined (whereas the modulus  $|\phi^{41}_{\hat{1}}|^2$  has definite non-zero vacuum expectation value of order  $J$ ) and thus averaged over in the path integral the vacuum expectation value of  $\phi^{41}_{\hat{1}}$  vanishes. Our idea is to gauge away the phase degrees of freedom of  $\phi^{41}_{\hat{1}}$  by choosing the unitary gauge in order to deal with this issue.

We define the real fields  $u, v$  such that

$$\phi^{41}_{\hat{1}} = \frac{1}{\sqrt{2}}(f + u + iv). \quad (3.28)$$

We will see later that an effective potential is generated such that the minimum of the potential is at a non-zero value of the real part of  $\phi^{41}_{\hat{1}}$ , which will be determined later. In (3.28) we have separated the vacuum expectation value  $f$  and the fluctuation around it,  $u$ , for convenience. The canonical conjugate momenta  $p_u$  and  $p_v$  satisfy

$$\pi_4^{\hat{1}} = \frac{1}{\sqrt{2}}(p_u - ip_v). \quad (3.29)$$

By requiring

$$v = 0, \quad (3.30)$$

together with the condition  $f + u \geq 0$ , we gauge away the phase degrees of freedom of  $\phi^{41}_{\hat{1}}$ . Other unitary gauge conditions we choose are

$$\phi^{41}_{\hat{i}'} = 0, \quad (3.31)$$

$$\phi^{4i'}_{\hat{1}} = 0. \quad (3.32)$$

which similarly eliminate the gauge freedom associated with  $(1, i')$  gauge transformations. One can prove, using the polar decomposition of matrices [66], that a configuration satisfying these gauge conditions can be obtained by performing a  $U(N) \times U(N)$  gauge transformation from any configuration of  $\phi^4$ .

We fix the remaining gauge freedom by imposing Coulomb gauge conditions. We decompose the fluctuation part of the gauge fields  $a_\alpha$  and  $\hat{a}_\alpha$  into

$$(a_\alpha)^i_j = \begin{bmatrix} z_\alpha & w_{\alpha j'} \\ (w_{\alpha i'})^* & (a_\alpha)^{i'}_{j'} \end{bmatrix}, \quad (\hat{a}_\alpha)^{\hat{i}}_{\hat{j}} = \begin{bmatrix} \hat{z}_\alpha & \hat{w}_{\alpha \hat{j}'} \\ (\hat{w}_{\alpha \hat{i}'})^* & (\hat{a}_\alpha)^{\hat{i}'}_{\hat{j}'} \end{bmatrix}, \quad (3.33)$$

defining the  $U(1)$  gauge fields  $z_\alpha$ ,  $\hat{z}_\alpha$  and the “W-bosons”  $w_{\alpha i'}$ ,  $\hat{w}_{\alpha \hat{i}'}$ . In the Hamiltonian and the Gauss law constraints, the fields  $z$  and  $\hat{z}$  appear frequently in the combinations

$$z^-_\alpha = z_\alpha - \hat{z}_\alpha, \quad (3.34)$$

$$z^+_\alpha = \frac{1}{2}(z_\alpha + \hat{z}_\alpha). \quad (3.35)$$

In terms of these fields, the Coulomb gauge conditions are

$$\text{div } z^+ = 0, \quad (3.36)$$

$$\text{div } a^{i'}_{j'} = 0, \quad (3.37)$$

$$\text{div } \hat{a}^{\hat{i}'}_{\hat{j}'} = 0. \quad (3.38)$$

The conditions (3.30)-(3.32) and (3.36)-(3.38), fix the gauge ambiguity (up to residual gauge transformations with constant parameters which will be discussed later)<sup>16</sup>. The next step is to solve the Gauss law constraints using the gauge fixing conditions. In this way the canonical momenta conjugate to the variables eliminated by the gauge conditions are rewritten in terms of the remaining physical degrees of freedom of the gauge-fixed theory. The variables to be eliminated using the Gauss law constraints are

$$p_v, \quad \pi_4^{\hat{1} i'}, \quad \pi^{41}_{\hat{j}'}, \quad \text{rot } z^-, \quad \text{rot } a^{i'}_{j'}, \quad \text{rot } \hat{a}^{\hat{i}'}_{\hat{j}'}. \quad (3.39)$$

The physical variables of the gauge-fixed theory are

$$\phi^{A' i}_{\hat{j}}, \quad \pi_{A' \hat{i} j}, \quad \psi^{A \hat{i}}_j, \quad u, \quad p_u, \quad \text{div } z^-, \quad \text{rot } z^+, \quad w_{\alpha i'}, \quad \hat{w}_{\alpha \hat{i}'}, \quad \phi^{4i'}_{\hat{j}'}, \quad \pi_4^{\hat{i}'}_{j'}, \quad (3.40)$$

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<sup>16</sup>Our gauge fixing conditions have some similarities to those used in [57, 59].

where we have introduced indices  $A', B' = 1, 2, 3$ . Once the Gauss laws are solved, one can compute the observables in the gauge-fixed theory, such as the Hamiltonian and various symmetry charges, by substituting the expression for the variables (3.39) into the original expression of these observables before fixing the gauge. We note that, for example,  $\text{div } z$  and  $\text{rot } z$  are canonically conjugate to each other in the following sense. Expanding the one-form field  $z_\alpha$  using the one forms constructed from the spherical harmonics,  $dY_{lm}$  and  $*dY_{lm}$  ( $l = 1, 2, \dots$ ), one can show that, up to a numerical factor depending on  $l$ , the coefficients in the expansion are canonically conjugate to each other. This expansion can be justified by using the standard Hodge decomposition theorem [70], which states that any differential form can be written as the sum of exact (rotationless), co-exact (divergenceless) and harmonic forms. A harmonic one-form on the sphere is necessarily singular and has the form of the gauge field for the Dirac monopole, which is excluded in our case, as discussed below (3.27). Hence one-form fields, such as  $z_\alpha$ , can be specified by giving  $\text{rot } z$  and  $\text{div } z$ .

The Gauss law constraints are non-linear and should be solved in an iterative manner in general. It is convenient to rewrite the Gauss law constraints (3.15), (3.16), by using (3.25), (3.26) and (3.27), as

$$\frac{k}{2\pi} F_{12}^{\text{BKG}} + \frac{k}{2\pi} \partial'_1 a_2 = i\phi^4 \pi_4 - i\pi^4 \phi_4 + \rho_W, \quad (3.41)$$

$$\frac{k}{2\pi} \hat{F}_{12}^{\text{BKG}} + \frac{k}{2\pi} \partial'_1 \hat{a}_2 = -i\phi_4 \pi^4 + i\pi_4 \phi^4 - \hat{\rho}_W, \quad (3.42)$$

where  $F_{12}^{\text{BKG}}$  and  $\hat{F}_{12}^{\text{BKG}}$  are field strengths for the background gauge fields,  $B_\alpha{}^i{}_j$  and  $B_\alpha{}^{\hat{i}}{}_{\hat{j}}$ , and  $\rho_W$  and  $\hat{\rho}_W$  are defined as

$$\rho_W{}^i{}_j = \left( i\phi^{A'} \pi_{A'} - i\pi^{A'} \phi_{A'} + \sin \theta \psi^T{}_A \psi^A + i \frac{k}{2\pi} [a_1, a_2] \right)^i{}_j, \quad (3.43)$$

$$\hat{\rho}_W{}^{\hat{i}}{}_{\hat{j}} = \left( i\phi_{A'} \pi^{A'} - i\pi_{A'} \phi^{A'} \right)^{\hat{i}}{}_{\hat{j}} - \sin \theta \psi^T{}_A{}^k{}_{\hat{j}} \psi^{A\hat{k}}{}_k - i \frac{k}{2\pi} [\hat{a}_1, \hat{a}_2]^{\hat{i}}{}_{\hat{j}}, \quad (3.44)$$

separating the contributions of  $\phi^4$  and  $\phi^{A'}$  ( $A' = 1, 2, 3$ ). The  $(1, 1)$  components of the operators  $\rho_W$  and  $\hat{\rho}_W$  may be thought of as charge densities for the fields,  $\phi^{A'}$  and  $\psi^A$ , and the “W-bosons”,  $w_{\alpha i'}$  and  $\hat{w}_{\alpha \hat{i}'}$ . We use the symbol  $\partial'$  to denote the covariant derivative defined in terms of the background gauge fields (3.27). For example

$$\partial'_\alpha \phi^{A'1}{}_{\hat{i}'} = \partial_\alpha \phi^{A'1}{}_{\hat{i}'} - iB_\alpha \phi^{A'1}{}_{\hat{i}'}, \quad (3.45)$$

$$\partial'_\alpha \phi^{A'i'}{}_{\hat{1}} = \partial_\alpha \phi^{A'i'}{}_{\hat{1}} + iB_\alpha \phi^{A'i'}{}_{\hat{1}}, \quad (3.46)$$

$$\partial'_\alpha \psi^{A\hat{1}}{}_1 = \partial_\alpha \psi^{A\hat{1}}{}_1, \quad (3.47)$$

$$\partial'_\alpha w_{\beta i'} = \partial_\alpha w_{\beta i'} - iB_\alpha w_{\beta i'} \quad (3.48)$$

$$\partial'_\alpha z_\beta = \partial_\alpha z_\beta. \quad (3.49)$$

We use underlined indices to indicate anti-symmetrisation (without any normalisation factors). For example

$$\partial'_1 a_2 = \partial'_1 a_2 - \partial'_2 a_1. \quad (3.50)$$

By adding and subtracting the  $(1, 1)$  components of the two Gauss law constraints (3.41) and

(3.42) under the unitary gauge condition (3.30)-(3.32), one obtains, respectively,

$$p_v = \frac{1}{f+u} \left( \frac{k}{2\pi} q \sin \theta + \frac{k}{2\pi} \partial_{\underline{1}} z_{\underline{2}}^+ - \frac{1}{2} \rho_W^1 \hat{1} + \frac{1}{2} \hat{\rho}_W \hat{1} \right), \quad (3.51)$$

$$\partial_{\underline{1}} z_{\underline{2}}^- = \frac{2\pi}{k} \left( \rho_W^1 \hat{1} + \hat{\rho}_W \hat{1} \right). \quad (3.52)$$

The right hand sides of these expressions are written only in terms of physical variables and thus  $p_v$  and  $\partial_{\underline{1}} z_{\underline{2}}^- \sim \text{rot } z^-$  are solved completely.

From the  $(1, i')$  and  $(i', j')$  components of the Gauss law constraints (3.41)-(3.42) under the unitary gauge condition (3.30)-(3.32), we obtain

$$\pi_4^{\hat{1}}{}_{i'} = -i \frac{\sqrt{2}}{f+u} \left( \frac{k}{2\pi} \partial_{\underline{1}}' w_{\underline{2}i'} - \rho_W^1{}_{i'} + i\pi^{41}{}_{\hat{j}'} \phi_4^{\hat{j}'}{}_{i'} \right), \quad (3.53)$$

$$\pi^{41}{}_{\hat{j}'} = i \frac{\sqrt{2}}{f+u} \left( \frac{k}{2\pi} \partial_{\underline{1}}' \hat{w}_{\underline{2}\hat{j}'} + \hat{\rho}_W \hat{1}_{\hat{j}'} - i\pi_4^{\hat{1}}{}_{i'} \phi^{4i'}{}_{\hat{j}'} \right), \quad (3.54)$$

$$\partial_{\underline{1}} a_{\underline{2}}^{i'}{}_{j'} = \frac{2\pi}{k} \left( i\phi^{4i'}{}_{\hat{k}'} \pi_4^{\hat{k}'}{}_{j'} - i\pi^{4i'}{}_{\hat{k}'} \phi_4^{\hat{k}'}{}_{j'} + \rho_W^{i'}{}_{j'} \right), \quad (3.55)$$

$$\partial_{\underline{1}} \hat{a}_{\underline{2}}^{\hat{i}'}{}_{\hat{j}'} = \frac{2\pi}{k} \left( -i\phi_4^{\hat{i}'}{}_{k'} \pi^{4k'}{}_{\hat{j}'} + i\pi_4^{\hat{i}'}{}_{k'} \phi^{4k'}{}_{\hat{j}'} - \hat{\rho}_W^{\hat{i}'}{}_{\hat{j}'} \right). \quad (3.56)$$

The right hand sides of these formulae are not written solely in terms of the physical variables. Hence to determine the fields  $\pi_4^{\hat{1}}{}_{i'}$ ,  $\pi^{41}{}_{\hat{j}'}$ ,  $\text{rot } a^{i'}{}_{j'}$  and  $\text{rot } \hat{a}^{\hat{i}'}{}_{\hat{j}'}$  it is necessary to proceed iteratively. The result is an infinite expansion for these fields in which the terms produced by each subsequent iteration contain a larger number of physical fields. For example, the right hand side of (3.53) contains  $\pi^{41}{}_{\hat{j}'} \phi_4^{\hat{j}'}{}_{i'}$ , which should be solved again using (3.54). This term contains quadratic and higher order terms in the fluctuation fields and, in the leading order, can be neglected. Also,  $\rho_W^1{}_{i'}$  contains the field  $z_{\alpha}$ , so part of it should be solved using (3.52).

The (iteratively) solved variables should be substituted into the original Hamiltonian (3.3) to obtain the Hamiltonian of the gauge-fixed theory. In general a number of iterations are needed to obtain all the terms in the Hamiltonian which are necessary to study a given process.

Eliminating the canonical momentum  $p_v$  through the Gauss law constraint (3.51) produces an effective potential for the real part of  $\phi^{41}{}_{\hat{1}}$ . Together with the mass term arising in radial quantisation, this results in a version of the Higgs mechanism inducing a vacuum expectation value for the real part of  $\phi^{41}{}_{\hat{1}}$ . This effect will play a crucial role in our analysis. The field  $p_v$  enters in the original Hamiltonian in the term

$$\int d\theta d\varphi \frac{1}{\sin \theta} \frac{1}{2} (p_v)^2. \quad (3.57)$$

Using (3.51) to express  $p_v$  in terms of the physical variables produces, among others, the term

$$\int d\theta d\varphi \frac{1}{32\pi^2} \frac{J^2}{(f+u)^2} \sin \theta. \quad (3.58)$$

This term gives an effective potential for the real part of the field  $\phi^{41}{}_{\hat{1}}$ , or the  $u$  field. It is analogous to the centrifugal potential in elementary mechanics. The  $u$  field feels another

potential, the mass term originating from the radial quantisation, present already in the original Hamiltonian (3.3). Hence the total potential for the  $u$  field is

$$\int d\theta d\varphi \sin \theta \left( \frac{1}{8}(f+u)^2 + \frac{1}{32\pi^2} \frac{J^2}{(f+u)^2} \right). \quad (3.59)$$

We fix the value of  $f$  by requiring that this potential be minimised at  $u = 0$ . It follows that

$$f = \sqrt{\frac{J}{2\pi}} = \sqrt{\frac{kq}{\pi}}, \quad (3.60)$$

which gives the vacuum expectation value of the real part of  $\phi^{41}_{\hat{1}}$  in the unitary gauge, or, equivalently, the vacuum expectation value of  $|\phi^{41}_{\hat{1}}|$ . The value of the potential at the minimum is  $J/2$ . This gives the energy of the perturbative vacuum in the gauge we are using,

$$\Delta = \frac{J}{2}, \quad (3.61)$$

reproducing the formula for the energy of the BPS ground state. Substituting  $\phi^{41}_{\hat{1}} = (f+u)/\sqrt{2}$  into the original Hamiltonian, one obtains various mass terms and interaction vertices containing factors of  $f$ . This introduces an explicit  $J$  dependence in the Hamiltonian which will play an important role in the following.

It turns out that in the gauge discussed above the states dual to the membrane fluctuations are created by the  $(1, 1)$  diagonal components of the various physical fields<sup>17</sup>. Furthermore, the  $(i', j')$  components are decoupled from the  $(1, 1)$  components at least in the first few orders in the approximation that should be valid in the large  $J$  regime, which will be discussed in section 3.3. Hence we should focus on the  $(1, 1)$  diagonal and  $(1, i')$  off-diagonal components of the physical fields. The gauge-fixed Hamiltonian quadratic in these components is derived substituting the expression for (3.39) obtained solving the Gauss law constraints into the original Hamiltonian (3.3),

$$\begin{aligned} H = & \frac{J}{2} + \int d\theta d\varphi \left[ \frac{1}{\sin \theta} \pi_{A'}^{\hat{1}} \pi_{A'}^{A'1} + \sin \theta g^{\alpha\beta} \partial_\alpha \phi^{A'1}_{\hat{1}} \partial_\beta \phi_{A'}^{\hat{1}} + \frac{1}{4} \sin \theta \phi^{A'1}_{\hat{1}} \phi_{A'}^{\hat{1}} \right. \\ & + \frac{1}{\sin \theta} \left( \pi_{A'}^{A'1} \pi_{A'}^{\hat{i}'} + \pi_{A'}^{A'i'} \pi_{A'}^{\hat{1}} \right) + \sin \theta g^{\alpha\beta} \left( \partial'_\alpha \phi^{A'1}_{\hat{i}'} \partial'_\beta \phi_{A'}^{\hat{i}'} + \partial'_\alpha \phi^{A'i'}_{\hat{1}} \partial'_\beta \phi_{A'}^{\hat{1}} \right) \\ & + \left( q^2 + \frac{1}{4} \right) \sin \theta \left( \phi_{A'}^{\hat{1}} \phi_{A'}^{A'i'} + \phi_{A'}^{A'1} \phi_{A'}^{\hat{i}'} \right) \\ & + \frac{1}{2 \sin \theta} p_u^2 + \frac{1}{2} \sin \theta u^2 + \frac{1}{2} \sin \theta g^{\alpha\beta} \left( \partial_\alpha u \partial_\beta u + f^2 z_\alpha^- z_\beta^- \right) - \frac{k}{2\pi f} u \partial_1 z_2^+ + \frac{k^2}{8\pi^2 f^2 \sin \theta} (\partial_1 z_2^+)^2 \\ & + \frac{2}{f^2} \left( \frac{k}{2\pi} \right)^2 \frac{1}{\sin \theta} \left( |\partial'_1 w_{2i'}|^2 + |\partial'_1 \hat{w}_{2\hat{i}'}|^2 \right) + \frac{f^2}{2} \sin \theta g^{\alpha\beta} \left( w_{\alpha i'} w_{\beta i'}^* + \hat{w}_{\alpha \hat{i}'} \hat{w}_{\beta \hat{i}'}^* \right) \\ & + \sin \theta \psi_{A'}^T \sigma^r \sigma^\alpha \partial_\alpha - 1 \psi^{A\hat{1}}_1 \\ & + \sin \theta \left\{ \psi_{A'}^T \sigma^r \sigma^\alpha \partial'_\alpha - 1 \psi^{A\hat{1}}_{i'} + \psi_{A'}^T \sigma^r \sigma^\alpha \partial'_\alpha - 1 \psi^{A\hat{i}'}_1 \right\} \end{aligned}$$

<sup>17</sup> We stress that a state with only  $(1, 1)$  excitations can be gauge-invariant, in the presence of a monopole operator, so long as it satisfies the Gauss law constraints. This is different from the situation in more familiar theories such as  $\mathcal{N} = 4$  Super Yang-Mills, where gauge-invariant operators are constructed taking the trace of products of fields and hence cannot be built out of single components of matrix fields.



$$\begin{aligned}
& -q \sin \theta \left( \psi^T_{4\hat{1}} \sigma^r \psi^{4\hat{1}'}_1 - \psi^T_{4\hat{1}'} \sigma^r \psi^{4\hat{1}}_{i'} \right) + q \sin \theta \left( \psi^T_{A'\hat{1}} \sigma^r \psi^{A'\hat{1}'}_1 - \psi^T_{A'\hat{1}'} \sigma^r \psi^{A'\hat{1}}_{i'} \right) \\
& - \frac{1}{4} \left( \rho_W^{11} - \hat{\rho}_W^{\hat{1}\hat{1}} \right) \Big].
\end{aligned} \tag{3.62}$$

To obtain this Hamiltonian no iteration is actually necessary and it is sufficient to drop higher order terms in the right hand side of (3.53)-(3.56). One important step involved in deriving the above expression is a partial integration

$$\int d\theta d\varphi \partial_1 z_2^+ = 0, \tag{3.63}$$

which is possible because the fluctuation field  $z$  does not contain a part proportional to the gauge field for the Dirac monopole, as discussed below (3.27). Strictly speaking, the term containing  $(z_\alpha^-)^2$  in the fourth line of (3.62) should be understood as signifying only the contribution from the divergence part of  $z^-$ . The rotation part of  $z^-$  should be rewritten using (3.52) and it produces only cubic or higher interaction terms.

From this Hamiltonian we have calculated the spectrum. From the structure of the covariant derivative (3.8) and the background field (3.27), it follows that the off-diagonal elements feel the background magnetic flux, whereas the diagonal elements do not feel the magnetic field, as exemplified in (3.45)-(3.49). It follows that the off-diagonal modes have to be expanded in terms of monopole spherical harmonics,  $Y_{qlm}$  [67] (and their generalisations to fermions and vectors), whereas the diagonal modes should be expanded in standard spherical harmonics,  $Y_{lm}$  (and their generalisations). An important point is that, for the monopole spherical harmonics, the quantum number  $l$  starts from  $q = J/(2k)$  (for spin 1/2 and 1 fields there are order 1 shifts), whereas for the standard spherical harmonics  $l$  of course starts from 0 (again with order 1 shifts for fields with spin). This effect (the order  $q$  shift of the lowest value of  $l$  due to magnetic flux) combines with the structure of the mass terms in (3.62), which are  $J$  dependent due to the Higgs effect. As a result, we find that the off-diagonal modes have large frequencies of order  $J$ , whereas the diagonal modes have small frequencies of order 1. We will call them fast (or high-energy) modes and slow (or low-energy) modes respectively.

This large separation between the two energy scales naturally leads to the idea that an approximation of the Born-Oppenheimer type should be applicable to our system. Namely, the fast modes should be integrated out and the effective theory thus obtained will have interactions which are suppressed by a power of the ratio between the two energy scales,  $1/J$ . We will discuss more in detail how this procedure should be implemented in our formulation in section 3.3. The slow modes ((1,1) components) represent physical states and their spectrum should be compared to the spectrum of fluctuations on the AdS side studied in section 2.2.

In this paper, we will only sketch the computation of the spectrum. We hope to present the details elsewhere. The results are summarised in tables 3 and 4. These spectra should be considered as the leading order result in the Born-Oppenheimer approximation.

For the mass spectrum of the slow modes of  $\phi^{A'}$ , the first and ninth lines in the Hamiltonian (3.62) contribute. By expanding  $\phi^{A'1\hat{1}}$  in spherical harmonics, one finds that the contribution of the first line to the conformal dimension is

$$\sqrt{\left(\frac{1}{2}\right)^2 + l(l+1)} = \frac{1}{2} + l. \tag{3.64}$$

The first term in the square root comes from the mass term, arising from the radial quantisation. The second term comes from the kinetic term associated with the Laplacian on  $S^2$ <sup>18</sup>. The contribution of the ninth line shifts the eigenfrequency by  $\pm 1/2$ .

The fast mode scalar spectrum is computed using an expansion in terms of monopole spherical harmonics. There is a mass term produced by the Higgs mechanism originating from the sextic potential proportional to  $f^4$ . The spectrum is integer (or half-integer) valued which is a consequence of the particular value of the mass term produced by the Higgs mechanism.

The slow and fast mode fermion spectra can be computed using the Clebsch-Gordan method. For our Hamiltonian obtained in part by the Higgs mechanism (containing a mass term for the fermions proportional to  $f^2$  coming from the original Yukawa term), the Clebsch-Gordan wave functions automatically give eigenmodes with rational eigenvalues. We note that in general for spin 1/2 fields in a monopole background, a diagonalisation of  $2 \times 2$  matrices is necessary after the Clebsch-Gordan procedure [71].

The slow mode vector fields mix with the scalar  $u$ . The computation of the spectrum is done by taking care of this mixing. For the one-form fields  $z^-$  and  $z^+$  we used an expansion in terms of  $dY_{lm}$  and  $*dY_{lm}$  respectively, since the associated physical fields are respectively rotationless and divergenceless.

The fast mode vector fields should be solved by expanding fields in a basis constructed from linear combinations of  $d'Y_{qlm}$  and  $*d'Y_{qlm}$ , where  $d'$  refers to the gauge covariant version of the exterior derivative associated with the background gauge field. Special care should be taken for the low-lying modes with  $l = q - 1$ , for which a special basis (not written in terms of  $d'Y$  and  $*d'Y$ ) is necessary. The basis we use is analogous to the monopole vector spherical harmonics of [72].

Since the Hamiltonian on the CFT side corresponds to  $\Delta$  whereas the Hamiltonian on the AdS side corresponds to  $\Delta - J_4/2$ , it is convenient to compute  $\Delta - J_4/2$  to compare the two sides. The calculation of the value of  $J_4$  for the various states is non-trivial for reasons associated with our choice of gauge which we discuss below. The charge  $J_4$  in the Hamiltonian formalism before gauge fixing is given by

$$J_4 = \int d\theta d\varphi \text{Tr} \left( i\phi^4 \pi_4 - i\pi^4 \phi_4 - \frac{1}{2} \sin \theta \psi^T{}_4 \psi^4 + \frac{1}{2} \sin \theta \psi^T{}_{A'} \psi^{A'} \right). \quad (3.65)$$

The charge in the gauge-fixed theory, obtained by substituting the solved variables, is

$$J_4 = J + \int d\theta d\varphi \left( -\frac{1}{2} \rho_W^1{}_1 + \frac{1}{2} \hat{\rho}_W^{\hat{1}}{}_{\hat{1}} \right) + \int d\theta d\varphi \text{Tr} \left( -\frac{1}{2} \sin \theta \psi^T{}_4 \psi^4 + \frac{1}{2} \sin \theta \psi^T{}_{A'} \psi^{A'} \right). \quad (3.66)$$

The  $J_4$  charges for various slow modes, both before and after gauge fixing, can be read off from these expressions and are summarised in table 2. The difference between the charges before and after gauge fixing may be understood from the following consideration. The original symmetry transformation associated with the  $J_4$  charge (before gauge fixing) does

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<sup>18</sup> Expressions such as (3.64), which can be schematically written as  $\sqrt{\text{const.} + (\text{mode no.})^2}$ , are reminiscent of the spectrum of BMN operators. However, there are important differences. In our case this formula arises as the leading order term in the Born-Oppenheimer approximation and it does not involve the membrane tension. In the BMN case, the corresponding formula is obtained resumming the expansion in the 't Hooft coupling and the ratio of the two terms under the square root involves the string tension.

Field	$J_4$ before gauge fixing	$J_4$ after gauge fixing
$\phi^{41}_{\hat{1}}$	1	0
$\phi^{A'1}_{\hat{1}}$	0	-1
$\psi^{4\hat{1}}_1$	$\frac{1}{2}$	$\frac{3}{2}$
$\psi^{A'\hat{1}}_1$	$-\frac{1}{2}$	$\frac{1}{2}$

Table 2:  $J_4$  charges of matter fields before and after gauge fixing. The difference is due to a compensating gauge transformation. The charges of the complex conjugate fields have opposite signs.

not preserve the gauge fixing condition (3.30). This implies the necessity of a compensating gauge transformation, resulting in a shift of the  $J_4$  charges in the gauge-fixed theory. In the comparison with the AdS side the  $J_4$  charge after gauge-fixing should be used.

In table 3 we have also shown  $\Delta - J_4/2$  for oscillators corresponding to various  $(1, 1)$  fields. The results are in complete agreement with the spectrum of fluctuations of the spherical membranes on the AdS side, summarised in table 1. We recall that the Hamiltonian on the AdS side corresponds to  $-P_0 - P_1 = 2(\Delta - J_4/2)/R$  because of (2.17) and (2.18). The agreement verifies the AdS/CFT correspondence in the leading order in a truly M-theoretic regime for non-BPS observables, which had not been studied before. The agreement also suggests the existence of an approximation scheme on the CFT side corresponding to the perturbative expansion on the AdS side discussed in section 2.3.

The spectrum of the slow modes has a simple interpretation in terms of the free field theory picture in section 3.1. As an example we consider a state in which one of the oscillators, with mode number  $l$ , associated with the field  $\phi^{A'1}_{\hat{1}}$  is excited. Having fixed the background (3.25), (3.26), and (3.27) means that we are considering states with fixed  $J_M = J$ . Hence in the picture of section 3.1 the state under consideration corresponds to a state in which the zero-mode of  $\phi^{41}_{\hat{1}}$  is excited  $J - 1$  times, and the oscillator  $\phi^{A'1}_{\hat{1}}$  is excited once. The excitation energy for the latter oscillator in the free field theory picture is given by (3.64), which corresponds to the bare dimension of the  $\phi^{A'}$  field with  $l$  derivatives acting on it. We note that in radial quantisation of a free scalar field theory, operators such as  $\partial^l \phi$  are mapped to states in which the oscillator with angular momentum quantum number  $l$  is excited once. The energy of this state in the free field theory picture is

$$\frac{1}{2} \times (J - 1) + \left( \frac{1}{2} + l \right) = \frac{J}{2} + l. \quad (3.67)$$

By comparing this with the energy of the ground state,  $J/2$ , we see that the excitation energy in this gauge should be  $l$ , in agreement with table 3. This gives a simple interpretation of the rationality of the energy spectrum (at tree level) on the AdS side in table 1, which might seem accidental from the point of view of the matrix model <sup>19</sup>.

<sup>19</sup> In our construction we use operators of the form  $\partial^l \phi$ , in the sense explained in this paragraph, to describe

Field	Label	$\Delta$	$\Delta - \frac{J_4}{2}$	Multiplicity
Scalars	$\phi^{A'1}_{\hat{1}}$	$l = 0, 1, \dots$	$\frac{1}{2} + l$	$3 \times (2l + 1)$
	$\phi_{A'}^{\hat{1}}{}_1$	$l = 0, 1, \dots$	$\frac{1}{2} + l$	$3 \times (2l + 1)$
Vectors (rot $z^+$ , div $z^-$ ) and $u$	$l = 0, 1, \dots$	$1 + l$	$1 + l$	$(2l + 1)$
	$l = 1, 2, \dots$	$l$	$l$	$(2l + 1)$
Fermions	$\psi^{A'\hat{1}}{}_1$	$j = \frac{1}{2}, \frac{3}{2}, \dots$	$\frac{3}{4} + j$	$3 \times (2j + 1)$
	$\psi_{A'}^1{}_{\hat{1}}$	$j = \frac{1}{2}, \frac{3}{2}, \dots$	$\frac{1}{4} + j$	$3 \times (2j + 1)$
	$\psi^{4\hat{1}}{}_1$	$j = \frac{1}{2}, \frac{3}{2}, \dots$	$\frac{1}{4} + j$	$1 \times (2j + 1)$
	$\psi_4^1{}_{\hat{1}}$	$j = \frac{1}{2}, \frac{3}{2}, \dots$	$\frac{3}{4} + j$	$1 \times (2j + 1)$

Table 3: Mass spectrum for slow modes

The gauge fixing conditions we use leave a residual gauge freedom corresponding to certain gauge transformations with constant parameters on the  $S^2$  time slice. This translates into the fact that the zero-mode part of the Gauss law constraint (3.52) is not solved. If one integrates both sides of (3.52) over  $\theta, \varphi$  the left hand side vanishes automatically by partial integration (since  $z^-$  does not have singularities associated with the Dirac monopole) and we obtain the constraint corresponding to the residual gauge symmetry,

$$0 = \int d\theta d\varphi \left( \rho_W^1{}_1 + \hat{\rho}_W^{\hat{1}}{}_{\hat{1}} \right). \quad (3.68)$$

This condition should be imposed on the states in the gauge-fixed theory. Similarly, from (3.55) and (3.56), we obtain the constraints

$$0 = \int d\theta d\varphi \left( i\phi^{4i'}_{\hat{k}'} \pi_4^{\hat{k}'}{}_{j'} - i\pi^{4i'}_{\hat{k}'} \phi_4^{\hat{k}'}{}_{j'} + \rho_W^{i'}{}_{j'} \right), \quad (3.69)$$

$$0 = \int d\theta d\varphi \left( -i\phi_4^{\hat{i}'}{}_{k'} \pi^{4k'}_{\hat{j}'} + i\pi_4^{\hat{i}'}{}_{k'} \phi^{4k'}_{\hat{j}'} - \hat{\rho}_W^{\hat{i}'}{}_{\hat{j}'} \right). \quad (3.70)$$

These constraints do not affect the  $(1, 1)$  slow modes, so the comparison to the AdS side is also not affected. However, the constraints (3.69) and (3.70) impose restrictions on the  $(1, i')$  and  $(i', j')$  excitations. We will elaborate further on this point in section 3.3.

We will now briefly discuss some aspects of  $\mathcal{N} = 6$  supersymmetry in the sector we are considering. We have fixed the form of the supersymmetry generators in the Hamiltonian

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the fluctuations around the ground state. This is in marked contrast with the BMN sector of the  $\mathcal{N} = 4$  supersymmetric Yang-Mills theory [23], where the operators involve only insertions satisfying  $\Delta - J = 1$ , such as scalar fields without any derivatives.

Field	Label	$\Delta$	Multiplicity
Scalars $(\phi^{A'1}_{\hat{i}'}, \phi_{A'\hat{1}_{i'}})$	$l = q, q + 1, \dots$	$\frac{1}{4} + l$	$(N - 1) \times 2 \times 3 \times (2l + 1)$
	$l = q, q + 1, \dots$	$\frac{3}{4} + l$	$(N - 1) \times 2 \times 3 \times (2l + 1)$
Vectors $(w_{\alpha i}, \hat{w}_{\alpha \hat{i}})$	$l = q - 1$	$-\frac{1}{4} + q$	$(N - 1) \times 2 \times (2q - 1)$
	$l = q, q + 1, \dots$	$\frac{3}{4} + l$	$(N - 1) \times 2 \times (2l + 1)$
	$l = q + 1, q + 2, \dots$	$\frac{1}{4} + l$	$(N - 1) \times 2 \times (2l + 1)$
Fermions $(\psi^A \hat{1}_{i'}, \psi_A \hat{1}_{i'})$	$j = q - \frac{1}{2}, q + \frac{1}{2}, \dots$	$\frac{3}{4} + j$	$(N - 1) \times 2 \times 4 \times (2j + 1)$
	$j = q + \frac{1}{2}, q + \frac{3}{2}, \dots$	$\frac{1}{4} + j$	$(N - 1) \times 2 \times 4 \times (2j + 1)$

Table 4: Mass spectrum for fast modes

formulation of the radially quantised ABJM theory (before gauge fixing) by requiring that they satisfy the correct superalgebra with the Hamiltonian (3.3) (at the classical level). The supercharges read

$$Q_{ABa} = X_{ABa} + \frac{1}{2} \epsilon_{ABCD} (Y_{CD}^* B)_a, \quad (3.71)$$

where  $B$  is the charge conjugation operator and

$$\begin{aligned}
X_{ABa} = \int d\theta d\varphi \text{Tr} & \left[ \psi^T \underline{A} a \pi \underline{B} - i (\psi^T \underline{A} \sigma_r \sigma^\alpha)_a D_\alpha \phi \underline{B} \sin \theta + \frac{i}{2} \psi^T \underline{A} a \phi \underline{B} \sin \theta \right. \\
& + i \frac{2\pi}{k} (\psi^T \underline{A} \sigma_r)_a \phi_C \phi^C \phi \underline{B} \sin \theta - i \frac{2\pi}{k} (\psi^T \underline{A} \sigma_r)_a \phi \underline{B} \phi^C \phi_C \sin \theta \\
& \left. - i \frac{4\pi}{k} (\psi^T \underline{C} \sigma_r)_a \phi \underline{A} \phi^C \phi \underline{B} \sin \theta \right], \quad (3.72)
\end{aligned}$$

$$\begin{aligned}
Y_{ABa} = \int d\theta d\varphi \text{Tr} & \left[ i (\psi^T \underline{A} \sigma_r)_a \pi \underline{B} - (\psi^T \underline{A} \sigma^\alpha)_a D_\alpha \phi \underline{B} \sin \theta + \frac{1}{2} (\psi^T \underline{A} \sigma_r)_a \phi \underline{B} \sin \theta \right. \\
& - \frac{2\pi}{k} \psi^T \underline{A} a \phi_C \phi^C \phi \underline{B} \sin \theta + \frac{2\pi}{k} \psi^T \underline{A} a \phi \underline{B} \phi^C \phi_C \sin \theta \\
& \left. + \frac{4\pi}{k} \psi^T \underline{C} a \phi \underline{A} \phi^C \phi \underline{B} \sin \theta \right]. \quad (3.73)
\end{aligned}$$

The superalgebra is

$$\begin{aligned}
[Q_{ABb}^{*T}, Q_{CDa}]_+ = & (\delta^A_C \delta^B_D - \delta^A_D \delta^B_C) (H \delta^b_a - L_i \sigma^{ib}_a) \\
& - (M^A_C \delta^B_D - M^A_D \delta^B_C - M^B_C \delta^A_D + M^B_D \delta^A_C) \delta^b_a, \quad (3.74)
\end{aligned}$$

where the Hamiltonian,  $H$ , given in (3.3) can be identified with the dilation operator. The flavour  $SU(4)$  symmetry generators,  $M^A_B$ , are

$$M^A_B = \tilde{M}^A_B - \frac{1}{4} \tilde{M}^C_C \delta^A_B, \quad (3.75)$$

$$\tilde{M}^A_B = \int d\theta d\varphi \operatorname{Tr} (i\phi^A \pi_B - i\pi^A \phi_B - \sin \theta \psi^T_B \psi^A) \quad (3.76)$$

and  $L_i$ ,  $i = 1, 2, 3$ , are the generators of the  $SO(3)$  rotational symmetry acting on the time-slice  $S^2$ ,

$$L_i = \int d\theta d\varphi \operatorname{Tr} \left[ \pi_A V^\alpha_i D_\alpha \phi^A + \pi^A V^\alpha_i D_\alpha \phi_A + \sin \theta \psi^T_A \left( iV^\alpha_i D_\alpha + \frac{1}{2} \sigma_i \right) \psi^A \right], \quad (3.77)$$

where

$$V^\alpha_i = \begin{bmatrix} V^\theta_1 & V^\theta_2 & V^\theta_3 \\ V^\varphi_1 & V^\varphi_2 & V^\varphi_3 \end{bmatrix} = \begin{bmatrix} \sin \varphi & -\cos \varphi & 0 \\ \cot \theta \cos \varphi & \cot \theta \sin \varphi & -1 \end{bmatrix}. \quad (3.78)$$

As consistency checks regarding supersymmetry and the identification of the vacuum in our gauge at the quantum level, we have verified the following. We have derived the gauge-fixed form of the supercharges  $Q_{AB}$  (by substituting the variables obtained by iteratively solving the Gauss law constraints) neglecting cubic and higher order terms in the physical fields. At this order  $Q_{AB}$  involves either terms linear in the fermionic fields or quadratic terms containing one fermionic and one bosonic field. It turns out that linear terms in the fermions are present in  $Q_{A'B'}$ , but not in  $Q_{A'4}$ . This implies that the vacuum is annihilated by the  $Q_{A'4}$  supercharges and thus it is 1/2 BPS. We have explicitly verified that the quantum version of the superalgebra (3.74) is satisfied at the level in which one only retains terms quadratic in the fast modes in all the generators in the superalgebra. Since the vacuum is annihilated by  $Q_{A'4}$ , this computation ensures that the vacuum energy (3.23) receives no leading order correction, *i.e.* there cannot be a shift from the zero-point energy.

### 3.3 Perturbation theory

In this section we discuss the approximation scheme which we propose to be relevant in the large  $J$  sector of the ABJM theory. We will present the general features including a diagrammatic representation of the approximation for various processes. We focus on contributions to the energy spectrum and discuss an estimate of the dependence on the parameters  $N$ ,  $k$  and  $J$  for some of the leading corrections. We will illustrate a specific contribution to the spectrum of scalar modes, which results in the same  $Nk/J^3$  dependence as the one-loop correction on the AdS side presented in section 2.3, provided that certain cancellations, which we expect in view of the large amount of supersymmetry in the ABJM theory, take place. It will be important to explicitly calculate the leading order corrections following the approach explained below and we hope to carry out such calculation in the future.

We focus on the case of a single non-zero GNO charge considered in the previous subsection. As already explained, the large  $J$  sector of the ABJM theory involves two types of modes: the slow modes (diagonal components of the fields), with eigenfrequencies of order 1, and the fast modes (off-diagonal components of the fields), with eigenfrequencies of order  $J$ . In general, if there are two types of degrees of freedom in a theory with very different

energy scales, one expects that a Born-Oppenheimer type approximation – or low-energy effective description – should be applicable. In the leading order of the Born-Oppenheimer approximation, one first solves the theory describing the fast modes treating the slow modes as fixed parameters. The result is used to construct the effective theory for the slow modes. The coupling of the slow modes in the resulting effective theory is suppressed by a power of the ratio of the two energy scales. The original application of the Born-Oppenheimer approximation was to the quantum theory of molecules in which an effective theory for the slow motion of the nuclei is obtained after studying the fast motion of the electrons in a potential produced by the nuclei with fixed positions.

In the context of the ABJM theory we are interested in, we expect the following features to be relevant for the emergence of a good approximation scheme for large  $J$ . First, the Abelian part of the action of the ABJM theory is essentially that of a free theory, since all couplings among the diagonal fields can be gauged away at least classically. Hence direct couplings between the slow modes associated with the  $(1,1)$  components of the fields, even if they are produced in the iteration process described around (3.53)-(3.56), should be unphysical. Therefore the interaction between slow modes should always involve the fast modes. Second, since the fast modes by definition have large quadratic terms in the action, we expect that their interactions can be treated perturbatively. Third, supersymmetry should play an important role in controlling the behaviour of quantum corrections. Even with the energy gap of order  $J$ , the potentially large zero-point energy could lead to large interactions between the slow modes through the fast modes. However, we expect the leading order contributions to cancel out for near-BPS states thanks to supersymmetry. The remaining terms should be suppressed by a power of  $1/J$ , which in the present case is the ratio between low and high energy scales.

These features are analogous to those encountered in the computation of scattering amplitudes for D-branes with a small relative velocity and a large impact parameter [73–77]. In the case of this system the potential vanishes for mutually commuting, diagonal, matrix coordinates of the D-branes, *i.e.* there are no direct couplings between the diagonal components. Interactions between the diagonal components (the positions of the D-branes) are only induced by the off-diagonal components (open strings stretched between the D-branes). Higher order couplings between off-diagonal modes are not the dominant contribution to the physics in the scattering of D-branes, because of the large mass of the open strings. Supersymmetry implies that the leading order terms in the interaction potential between the diagonal modes mediated by the off-diagonal modes cancel out<sup>20</sup>. The remainder is the small interaction between D-branes suppressed by a ratio of powers of the small relative velocity and the large impact parameter.

A similar approach based on the existence of very different energy scales is familiar in the context of quantum field theory. In this case one performs the path integral only over

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<sup>20</sup> Actually, this cancellation was discussed before the advent of D-branes in the matrix model context from the membrane point of view in [78]. The cancellation implies that the matrix model has a continuous energy spectrum. More precisely, it implies the existence of states with arbitrarily small energy. This was incorrectly interpreted as signifying an instability of membranes. The interpretation was revised in recent years [46] after the D0-brane picture of [6]: the existence of states with arbitrarily small energy only means that the matrix model is a theory which describes multiple membranes, not a single membrane. Equivalently, the matrix model is a second-quantised rather than a first-quantised theory of membranes and as such it naturally has a continuous spectrum.



the high-energy degrees of freedom (the fast modes) to find an effective theory governing the dynamics of the low-energy degrees of freedom (the slow modes). This “integrating out” procedure to compute the effective action has a simple realisation in terms of Feynman diagrams as explained for example in [79]. The vertices in the effective action are obtained from Feynman diagrams in which all internal lines correspond to fast modes and the external lines only involve slow modes. We will use this procedure to construct the low-energy effective description for the ABJM theory in the large  $J$  regime.

A simple way of constructing the path integral of the gauge-fixed ABJM theory discussed in section 3.2 is to use a phase space formulation, in which the functional integration is performed over both the canonical variables and their conjugate momenta. For instance, for a complex scalar field in Euclidean signature, the Boltzmann factor is

$$\exp \int dt \int d^2x \left( i\pi \frac{\partial}{\partial t} \phi + i\pi^* \frac{\partial}{\partial t} \phi^* - \mathcal{H} \right), \quad (3.79)$$

where  $\mathcal{H}$  is the Hamiltonian density.

In our gauge the slow modes are the  $(1, 1)$  components of the various fields and the fast modes are the  $(1, i')$  components. We will comment on the role of  $(i', j')$  components later in this subsection. It is in principle reasonable to classify high momentum modes of the  $(1, 1)$  components as fast modes as well, since their eigenfrequencies are of the same order as those of the  $(1, i')$  components. From this point of view, we obtain a natural UV cut-off for the  $(1, 1)$  slow modes which is reminiscent of the UV cut-off arising on the AdS side as a consequence of the fact that we consider matrices of large but finite size, as noted below (2.51). The difference between the two prescriptions, *i.e.* whether one treats the high momentum  $(1, 1)$  fields as fast or slow, may affect the technical details of the calculation, but should not produce any difference in the final low-energy observables.

In order to obtain the propagators of the fast modes, we expand them in a basis constructed from the monopole spherical harmonics,  $Y_{qlm}$ . For example for the scalar fast modes we use

$$\phi^{A'}_{\hat{i}'} = \sum_{l=q}^{+\infty} \sum_{m=-l}^{+l} \int d\omega \left( \phi^{A'}_{\hat{i}'} \right)^{lm\omega} Y_{qlm} e^{i\omega t}, \quad (3.80)$$

$$\pi_{A'}^{\hat{i}'} = \sum_{l=q}^{+\infty} \sum_{m=-l}^{+l} \int d\omega \left( \pi_{A'}^{\hat{i}'} \right)_{lm\omega} (Y_{qlm} e^{i\omega t})^* \sin \theta. \quad (3.81)$$

In this subsection we omit the colour index 1 or  $\hat{1}$  from the fast modes, for brevity. The propagators can be computed in a standard manner. For instance one finds

$$\left\langle \left( \phi_{A'}^{\hat{i}'} \right)_{lm\omega} \left( \phi_{\hat{j}'}^{B'} \right)^{l'm'\omega'} \right\rangle = \frac{1}{2\pi} \frac{1}{\left( \omega - \frac{i}{4} \right)^2 + \left( l + \frac{1}{2} \right)^2} \delta(\omega - \omega') \delta^{B'}_{A'} \delta^{\hat{i}'}_{\hat{j}'} \delta^{l'}_l \delta^{m'}_m. \quad (3.82)$$

Since we work in the phase space path integral formalism, there are also other propagators for the scalar fields, *i.e.*  $\langle \phi\pi \rangle$ ,  $\langle \pi\phi \rangle$  and  $\langle \pi\pi \rangle$ . The variables  $\omega$ ,  $l$ , and  $m$  can be considered as the components of 3-momentum on the space-time  $S^2 \times \mathbb{R}$ . For each loop, one has the integration over  $\omega$  and summation over  $l$  and  $m$ . The index  $l$  is summed from  $q$  (with order 1 shifts for fields with non-zero spin) to infinity and the index  $m$  runs from  $-l$  to  $+l$ .

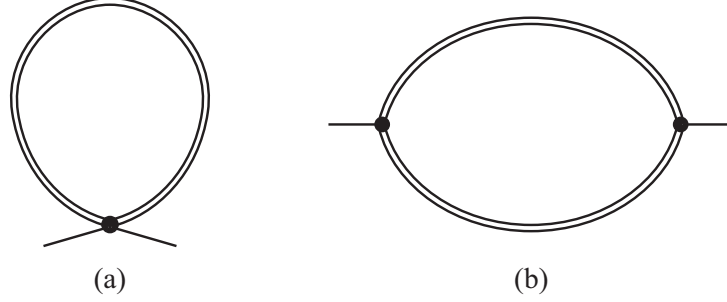


Figure 1: One-loop contributions to slow mode quadratic term in the effective action. Single lines correspond to  $(1, 1)$  slow modes and double lines to  $(1, i')$  fast modes. These processes are expected to be subleading because of cancellations due to supersymmetry.

The vertices can be read off from the gauge-fixed Hamiltonian. In order to obtain the interaction terms, such as for instance the cubic and quartic vertices, it is necessary to iterate equations (3.53)-(3.56) further than has been done in section 3.2 for the quadratic part of the Hamiltonian. Notice that because of the structure of the colour indices, all vertices contain an even number of fast modes.

A possible correction to the leading order spectrum considered in section 3.2 comes from the processes associated with the Feynman diagrams depicted in figure 1. In all the diagrams in this section we represent the  $(1, 1)$  slow modes with single lines and the  $(1, i')$  fast modes with double lines. These processes produce a direct radiative correction to the slow-slow term in the low-energy effective action. Even with the requirement that the interaction vertices in figure 1 should be only of slow-fast-fast and slow-slow-fast-fast kind, there is a very large number of contributions to both types of diagrams. An example of fast-fast-slow-slow vertex is

$$\int d^3x \left(\frac{f}{k}\right)^2 \sin \theta \phi^{A'}_{\hat{i}} \phi^{B'1}_{\hat{i}'} \phi^{B'1}_{\hat{1}} \phi^{A'1}_{\hat{1}}, \quad (3.83)$$

where we omitted purely numerical factors, but we kept the  $k$  dependence. This term, which is produced by the Higgs mechanism from the sextic scalar potential, is relevant for the diagram in figure 1 (a) with two scalar slow modes as the external lines. The behaviour of this contribution (at leading order) can be computed using (3.83) and (3.82). We get

$$\left(\frac{f}{k}\right)^2 \times (N-1) \times \sum_{l=q}^{\infty} \sum_{m=-l}^l \int d\omega \frac{1}{\omega^2 + l^2} \sim \frac{Nf^2}{k^2} \sum_{l=q}^{\infty} \sum_{m=-l}^l \frac{1}{l}. \quad (3.84)$$

This expression diverges linearly. This divergence should be cancelled by other contributions to the spectrum at the same order. Among the additional corrections which can contribute to the cancellation are diagrams of the type in figure 1 (a) with different four-point vertices and other (vector and fermion) internal lines. Moreover one has to consider the “vacuum polarisation” diagrams of the type in figure 1 (b), again with all possible internal lines. All these contributions have the same dependence on the parameters,  $N$ ,  $k$  and  $J$ , as (3.84). Finally, although the ABJM theory is believed to be UV finite, there may be a residual divergence after combining all diagrams, which needs to be absorbed into an unphysical – and generally gauge-dependent – wave function renormalisation.

It is important that the result of the loop integrals, or more precisely of the sums over  $l$  and  $m$  and the integral over  $\omega$ , is always organised in an expansion in powers of  $q^{-1}$  and the parameters  $J$  or  $k$  never appear explicitly. Assuming there is an  $n_0$ -fold cancellation as a result of combining all the above contributions in figure 1 and potential unphysical counter terms (with  $n_0 = 1$  meaning cancellation of the leading order contribution,  $n_0 = 2$  cancellation of the leading and next-to-leading order contributions *etc.*) we obtain

$$\frac{Nf^2}{k^2}q^{1-n_0}. \quad (3.85)$$

For  $n_0 = 1$  the sum is generically logarithmically divergent and we expect  $n_0 \geq 2$ . Rewriting (3.85) in terms of  $N$ ,  $k$  and  $J$ , we obtain

$$Nk^{n_0-3}J^{2-n_0}. \quad (3.86)$$

This expression cannot give rise to the same dependence on  $N$ ,  $k$  and  $J$  found on the AdS side, *i.e.*  $Nk/J^3$ , for any value of  $n_0$ . This leads us to conjecture that either  $n_0$  is sufficiently large,  $n_0 \geq 6$ , so that this type of correction is negligible compared to the expected leading order correction of order  $Nk/J^3$ , or the various contributions completely cancel out. We note that the estimate (3.85) is the leading order term and there are also higher order terms in the expansion in terms of  $q^{-1}$ .

We expect the leading order correction to the spectrum to come from the processes associated with the Feynman diagrams depicted in figure 2. These are one-loop diagrams in

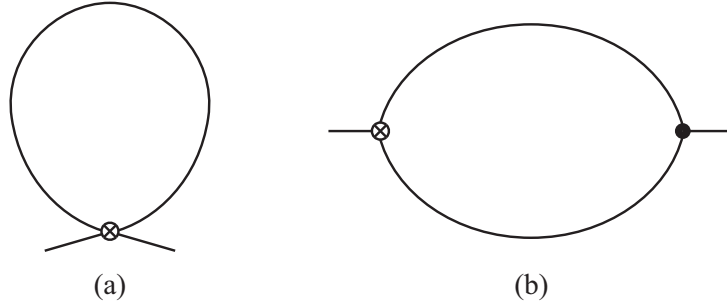


Figure 2: Correction to the spectrum at one-loop level in the low-energy theory for the  $(1,1)$  slow modes. Crossed white dots are effective vertices induced by one-loop diagrams in the fast modes. Black dots represent genuine vertices for the slow modes. These diagrams are expected to give the leading correction of order  $Nk/J^3$ .

the low-energy theory for the slow modes involving effective vertices obtained integrating out fast mode loops. We denote such effective vertices by crossed white dots. Black dots indicate vertices present in the original gauge-fixed Hamiltonian.

Let us focus for definiteness on corrections to the scalar spectrum. In this case the external lines in figure 2 are  $(1,1)$  components of scalar fields. In diagram (a) the quartic effective vertex couples the two scalars to two other slow mode fields which, depending on the type of loop, can be two scalars, two vectors or two fermions. The corresponding quartic effective vertices receive contributions from all diagrams in the full theory with four external slow-mode lines and internal fast-mode lines. Those relevant for the corrections to the spectrum

at order  $Nk/J^3$  involve a single fast-mode loop and are depicted in figure 3. To determine the vertex relevant for each type of slow mode loop in figure 2 (a) one has to compute all the contributions to four-point functions from the diagrams in figure 3 where two external lines are slow mode scalars and the other two are slow mode scalars, vectors or fermions respectively. After performing the loop integrals, one can extract the quartic effective vertex for the slow modes. It is straightforward to estimate the dependence on  $N, k$  and  $J$  for the diagrams in figure 3. For instance, for a diagram of type (c), in which both vertices are given

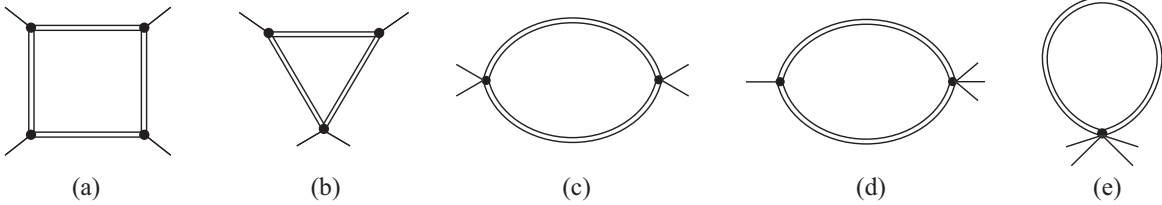


Figure 3: One-loop contributions to a quartic effective vertex in the low-energy effective action.

by (3.83) and all internal and external lines are scalar fields, the leading order contribution is

$$\left(\frac{f^2}{k^2}\right)^2 \times (N-1) \times \sum_{l=q}^{\infty} \sum_{m=-l}^l \int d\omega \left(\frac{1}{\omega^2 + l^2}\right)^2 \sim \frac{NJ^2}{k^4} \sum_{l=q}^{\infty} (2l+1) \frac{1}{l^3} \quad (3.87)$$

The internal loops in figure 3 can correspond to scalars, fermions or vectors. The different contributions can be analysed in a similar fashion and they all lead to the same dependence on the parameters in the scalar quartic effective vertex. Assuming again  $n_0$ -fold cancellations among these diagrams and possible counter terms, the behaviour we find is

$$\frac{NJ^2}{k^4} \sum_{l=q}^{\infty} (2l+1) \frac{1}{l^{3+n_0}} \sim \frac{NJ^2}{k^4} \frac{1}{q^{1+n_0}} \sim \frac{Nk^{n_0-3}}{J^{n_0-1}}. \quad (3.88)$$

For  $n_0 = 4$  this expression gives  $Nk/J^3$ . This is the same as the weight of the quartic fluctuations about the fuzzy sphere vacuum relative to the quadratic terms in the matrix model Hamiltonian studied in section 2.3. One contribution of the type we are describing corresponds to the two-loop diagram in the full theory shown in figure 4.

In the above derivation of the estimates (3.87) and (3.88) we considered the case of an internal slow mode scalar loop in figure 2 (a). Diagrams with an internal fermion or vector loop can also be shown to contribute to the two-point function at the same order  $Nk/J^3$ , again assuming appropriate cancellations. A method to obtain power counting estimates which can be applied to generic diagrams will be outlined later in this subsection. Another class of leading order corrections to the slow mode spectrum is associated with diagrams of the type depicted in figure 2 (b). This is a one loop diagram in the low-energy theory with one effective vertex and one genuine cubic vertex coupling slow modes. The leading contribution to the effective cubic vertex is generated by the fast mode one-loop diagrams depicted in figure 5. Again the loop in figure 2 (b) can involve scalar, vector or fermion slow mode fields. For each case a suitable cubic effective vertex is determined from diagrams of the type in figure 5 with the appropriate external slow mode lines.

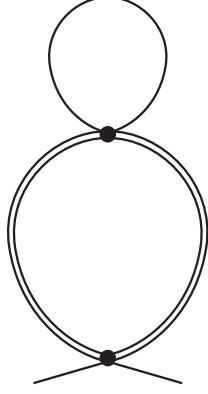


Figure 4: A typical leading order correction to the slow mode spectrum

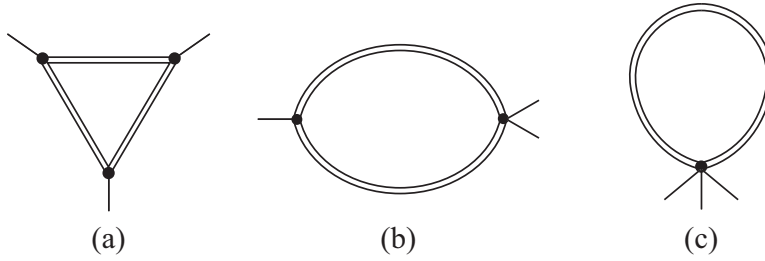


Figure 5: One-loop contributions to a cubic effective vertex in the low-energy effective action.

When combining all the contributions to the two types of diagrams in figure 2 to extract the correction to the spectrum we expect further cancellations in the slow mode loops, so that no extra powers of  $q$  are produced and the final correction to the two-point function is of order  $Nk/J^3$ . These cancellations at the level of the slow modes are analogous to the cancellations observed in the pp-wave matrix model, which ensure that the sums over intermediate states do not produce extra factors of  $J/k$ .

The corrections to the vector and fermion slow mode spectrum can be studied in a similar way. We verified by a similar power-counting that the leading non-zero contributions can come from two-point functions of the type in figure 2 with vector or fermion external lines, assuming again appropriate cancellations.

In the computation of the leading order corrections to the spectrum involving the diagrams in figures 2 and 3, the  $(i', j')$  components of the fields are indeed unimportant and decouple from the physics of  $(1, 1)$  modes. From the integrated Gauss law constraints, (3.68)-(3.70), the excitation of  $(i', j')$  components corresponds to gauge invariant operators constructed from the trace of products of matter fields and their complex conjugates, *e.g.*  $\phi^{A'}$  and  $\phi_{A'}$ . The existence of this type of state in the ABJM theory is expected from considerations on the gravity side. A configuration associated with a combination of  $(1, 1)$  and  $(i', j')$  field components corresponds to a “multi-particle” state in the pp-wave matrix model involving spherical membranes with large  $J_4$  and  $J_M$  as well as gravitons or other particles with vanishing  $J_4$  and  $J_M$ . It is natural to expect a suppression in the coupling between the membranes

and these extra particles because of the large difference in momentum. This supports our expectation that the coupling of the  $(i', j')$  components to the physical  $(1, 1)$  slow modes should be weak in the large  $J$  sector. It is not straightforward, at this stage, to determine the role of the  $(i', j')$  fields at higher orders in the Born-Oppenheimer approximation. Below we will discuss a different perspective in which these degrees of freedom can be understood in a more straightforward fashion. We also notice that as a consequence of the integrated Gauss law constraints single  $(1, i')$  fast modes cannot be excited and they should always appear in pairs. This is related to the fact that all vertices contain an even number of fast modes.

The Born-Oppenheimer approach and the description of the physics in terms of a low-energy effective action provide a very natural framework in which the emergence of a good approximation in the large  $J$  sector of the ABJM theory is motivated by physical considerations. However, from a practical point of view it may be technically simpler to compute the quantum corrections to the spectrum using the full theory, *i.e.* studying contributions to the two-point functions of the  $(1, 1)$  modes from all Feynman diagrams without the restriction that the internal lines be  $(1, i')$  fields. Power counting arguments similar to those presented above can be applied in this case as well. The finiteness of the ABJM theory plays an essential role in the power counting analysis. Since there is no dimensional transmutation in a finite theory, one gets a  $q$  dependence even for massless propagators because of the presence of massive propagators in the diagrams. Although at first sight there is no reason to expect the interactions among the  $(i', j')$  fields to be suppressed, these fields inherit the suppression by negative powers of  $q$  from the fast modes.

Computations in the full theory can be described in terms of Feynman diagrams using the standard double line notation, in which index loops represent sums over the colour index  $i$  (or  $\hat{i}$ ) from 1 to  $N$ . To each index loop in a diagram one has to assign either an index taking the value 1 or a primed index taking the values  $2, \dots, N$ . This assignment determines which of the internal lines are of  $(1, 1)$ ,  $(1, i')$  or  $(i', j')$  type. For each of these internal lines one should use the appropriate propagator in the gauge we have fixed. In a diagram in which  $p$  of the index loops carry a primed index taking values  $2, \dots, N$ , the colour contractions produce a factor  $(N - 1)^p$ . The integer  $p$  ranges from 0 to the number of index loops in the diagram. The latter equals the total number of loops for planar diagrams and decreases with the degree of non-planarity. As a result, we are using a large  $N$  expansion which is different from the standard planar expansion. For example the first subleading term in our expansion receives contributions from planar diagrams for which one index loop carries the index 1 as well as from the leading non-planar diagrams with all index loops carrying primed indices.

Using a method similar to the standard power counting argument one can show that the dependence on  $N$ ,  $k$  and  $q$  of a  $L$ -loop correction to the  $(1, 1)$  slow-mode spectrum from arbitrary diagrams is given by

$$k^{-L}(N - 1)^p q^{D-n}. \quad (3.89)$$

Here  $D$  is the mass dimension of the coefficient of the quadratic term in the action for the field corresponding to the two external lines. For example in the case of a scalar  $\phi^2$  term one has  $D = 2$ , for a fermion  $\psi^2$  term  $D = 1$ . As noted above, the summations and integrals over loop variables give rise to an expansion in inverse powers of  $q$ . The integer  $n$  in (3.89) specifies the order in this expansion. Because of the cancellations we expect the integer  $n$  to be greater than or equal to a certain positive integer,  $n_0$ , which is the order of the cancellation used in (3.85), (3.86) and (3.88). The integer  $p$  denotes the number of index loops which are

assigned the values  $2, \dots, N$  as explained above. By trivial rearranging of terms  $(N - 1)$  in the above formula can be replaced by  $N$ . We will do this for simplicity below. In order to derive this power counting estimate it is convenient to rescale the variables  $\phi$ ,  $\pi$  and  $\psi$  in such a way that the action functional in the path integral can be written with a common overall factor of  $k$ .

We do not yet have a complete understanding of the systematics of the perturbative expansion at higher orders: the power counting argument explained above does not take into account cancellations expected due to the large amount of supersymmetry in the ABJM theory. For the scattering of D0 branes in the context of the matrix model of M-theory – which, as mentioned earlier, has some close analogies to the case we are considering – the general structure of the expansion in terms of powers of the relative velocity and the impact parameter was discussed in [80].

We conclude this section with a few observations comparing the expansion (3.89) and the results obtained on the gravity side from the pp-wave matrix model. On the AdS side, for the membrane states we have considered in section 2, there are two coupling constants,  $Nk/J^3$  and  $J^2/Nk$ , which are associated with the loop expansion and the corrections to the pp-wave approximation. Hence, for processes in which these states are relevant, the parameters  $N$  and  $k$  should always appear with the same power. From (3.89) we find that this is achieved if the parameter  $n$  is

$$n = L + p + D. \quad (3.90)$$

Let us focus on these contributions <sup>21</sup>. Substituting back into (3.89) the order estimate becomes

$$(Nk)^p J^{-L-p}. \quad (3.91)$$

Rewriting this in terms of the two coupling constants on the AdS side we obtain

$$\left(\frac{Nk}{J^3}\right)^{L-p} \left(\frac{J^2}{Nk}\right)^{L-2p}. \quad (3.92)$$

We note that by definition  $0 \leq p \leq L$  and for smaller  $p$  there are more  $(1, 1)$  propagators and the number of possible Feynman diagrams decreases.

The leading order term of the pp-wave approximation we have considered in section 2 corresponds to

$$p = \frac{L}{2}. \quad (3.93)$$

In this case (3.92) reduces to

$$\left(\frac{Nk}{J^3}\right)^{\frac{L}{2}}, \quad (3.94)$$

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<sup>21</sup> The fact that  $N$  and  $k$  always appear in the combination  $Nk$  in the corrections discussed in section 2 has a simple interpretation. The curvature radius of the  $\text{AdS}_4 \times S^7/\mathbb{Z}_k$  background is written only in terms of  $Nk$ , see (2.3). It is natural that local fluctuations of the membranes only feel the curvature and do not detect the effect of the  $\mathbb{Z}_k$  quotient dividing the  $S^7$  into  $k$  pieces. However, in general there are other corrections some of which we expect to depend separately on  $N$  or  $k$ . Hence the existence of terms in (3.89) which do not satisfy (3.90) does not necessarily lead to a contradiction. The appearance of the combination  $Nk$  based on considerations on the gravity side and its implications for properties of the ABJM theory were discussed in [81].



*i.e.* the power of  $Nk/J^3$  in the expansion coincides with half the number of loops. This in particular implies that, for each given order in the expansion in terms of  $Nk/J^3$ , there is only a finite number of diagrams contributing and hence only a finite set of vertices in the gauge-fixed Hamiltonian are necessary. The processes corresponding to the leading order contribution of order  $Nk/J^3$  depicted in figure 2 satisfy  $L = 2, p = 1$ , as it should be.

The corrections to the pp-wave approximation come with positive powers of  $J^2/Nk$  and thus correspond to diagrams satisfying

$$p < \frac{L}{2}. \quad (3.95)$$

Therefore if no terms with  $p > L/2$  arise from perturbative calculations (3.92) has a straightforward interpretation as dual to the double expansion – associated with loops and corrections to the pp-wave approximation – discussed below (2.65). The explanation of terms with  $p > L/2$  in (3.92) is less clear, since they are singular for vanishing  $J^2/Nk$ . However, an infinite series in negative powers of  $J^2/Nk$  may yield a finite non-singular result, which might correspond to a non-perturbative correction to the pp-wave approximation in the matrix model.

In general the form of the low-energy effective action or Hamiltonian of a theory is strongly constrained by symmetry requirements. This is especially the case for supersymmetric theories, see [82] for a review. For the D0-brane scattering in the matrix model of M-theory this has been studied extensively, see, for example, [83] and references therein. At the end of section 3.2 we have discussed some aspects of the supersymmetry algebra of the ABJM theory in the formalism used in this paper. It would be very interesting to study the restrictions imposed by supersymmetry on the structure of the effective action and on the spectrum.

## 4 Multiple membrane case

As discussed in section 2.1, general zero energy configurations in the matrix model, obtained solving (2.31), correspond to concentric fuzzy spheres with angular momenta  $J_{(i)}$  in  $S^7$  and extending in  $\text{AdS}_4$  with radii  $r_{(i)} \approx J_{(i)}/2\pi TR^2$ . In order to be able to treat these configurations perturbatively in the pp-wave approximation, the individual  $J_{(i)}$ 's should satisfy the condition (2.63). The multi-membrane vacua correspond to states in the ABJM theory characterised by GNO charges  $q_{(i)} = J_{(i)}/2k$ , satisfying

$$\sum_i 2q_{(i)} = \frac{J}{k}. \quad (4.1)$$

For simplicity in this section we will focus on the case of two non-zero GNO charges,  $q_{(1)} = J_{(1)}/2k$ ,  $q_{(2)} = J_{(2)}/2k$ ,  $q_{(3)} = \dots = q_{(N)} = 0$ , but the following considerations have a straightforward generalisation to cases with more membranes and GNO charges.

From the definition of the covariant derivative (3.8) it follows that in general the  $(i, j)$  component of a bi-fundamental field has magnetic charge  $q_{(i)} - q_{(j)}$ . Therefore in the presence of two non-zero GNO charges,  $q_{(1)}$  and  $q_{(2)}$ , we have the following situation, which, for concreteness, we illustrate in the case of the scalar fields,  $\phi^{Ai}_{\hat{j}}$ . The other matter fields have a similar structure. The (block) diagonal components – consisting of two  $1 \times 1$  blocks,  $\phi^{A1}_{\hat{1}}$  and  $\phi^{A2}_{\hat{2}}$ , and a  $(N-2) \times (N-2)$  block,  $\phi^{Ai'}_{\hat{j}'}$ , ( $i', \hat{j}' = 3, \dots, N$ ) – have zero charge, as in the

single membrane case. There are two  $1 \times (N-2)$  blocks,  $\phi^{A1}_{\hat{i}'}$  and  $\phi^{A2}_{\hat{i}'}$ , and two  $(N-2) \times 1$  blocks,  $\phi^{Ai'}_{\hat{1}}$  and  $\phi^{Ai'}_{\hat{2}}$ , whose components carry charges  $\pm q_{(1)}$  and  $\pm q_{(2)}$ . Finally the  $\phi^{A1}_{\hat{2}}$  and  $\phi^{A2}_{\hat{1}}$  components have charges  $\pm(q_{(1)} - q_{(2)})$ . Thus the scalar fields are decomposed as

$$\phi^{Ai}_{\hat{j}} = \begin{bmatrix} \phi^{A1}_{\hat{1}} & \phi^{A1}_{\hat{2}} & \phi^{A1}_{\hat{j}'} \\ \phi^{A2}_{\hat{1}} & \phi^{A2}_{\hat{2}} & \phi^{A2}_{\hat{j}'} \\ \phi^{Ai'}_{\hat{1}} & \phi^{Ai'}_{\hat{2}} & \phi^{Ai'}_{\hat{j}'} \end{bmatrix}. \quad (4.2)$$

From the discussion in the previous section, one would expect that all the off-diagonal components should be identified as fast modes and integrated out, while the  $(1,1)$  and  $(2,2)$  diagonal components should correspond to slow modes associated with membrane excitations. However, the condition (2.63) for  $J_{(1)}$  and  $J_{(2)}$  implies  $q_{(1)} \gg 1$  and  $q_{(2)} \gg 1$ , but in general it is possible to have  $q_{(1)} - q_{(2)} \sim O(1)$  (and even  $q_{(1)} - q_{(2)} = 0$ )<sup>22</sup>. In this case the Born-Oppenheimer approximation requires that the  $(1,2)$  and  $(2,1)$  components of the ABJM fields be treated as slow modes, since they feel a magnetic charge  $q_{(1)} - q_{(2)}$  and therefore their expansion in monopole spherical harmonics starts with quantum number  $l = |q_{(1)} - q_{(2)}| \sim O(1)$ .

The simplest states in this class of  $(1,2)$  slow modes correspond to excitations of the complex scalar fields  $(\phi^{A'})^{i}_{\hat{j}}$ ,  $A' = 1, 2, 3$  with  $i = 1, \hat{j} = 2$  or  $i = 2, \hat{j} = 1$ . The spectrum for these states can be computed in a gauge similar to that used in section 3.2 in which we set  $\phi^{A1}_{\hat{2}} = 0$  and  $\phi^{A2}_{\hat{1}} = 0$ . The calculation is very similar to that for the  $(1, i')$  fast modes, for which the spectrum is given in table 4. For these  $(1,2)$  scalars the resulting spectrum is

$$\Delta - \frac{J_4}{2} = \frac{1}{2} + l, \quad (4.3)$$

where  $l = |q_{(1)} - q_{(2)}|, |q_{(1)} - q_{(2)}| + 1, \dots$  and the multiplicity is  $2 \times 3 \times (2l + 1)$ , with the factor of 2 due to the fact that the fields are complex.

Vector and fermion excitations contain extra slow modes as well. These can be studied in a similar fashion, however, their analysis requires a lengthier computation which we have not completed and thus we will not present the details here.

The generalisation to the case of three or more non-zero GNO charges is straightforward. For example in the case of three GNO charges,  $q_{(1)} \sim q_{(2)} \sim q_{(3)}$ , there are extra slow modes associated with the  $(1,2)$ ,  $(1,3)$ ,  $(2,3)$  and  $(2,1)$ ,  $(3,1)$ ,  $(3,2)$  components of the fields.

Having found this new set of low-energy excitations in the ABJM theory, we should be able to identify dual configurations on the matrix model side, corresponding to excitations of the multi-membrane vacua. Focussing again on the two-membrane case, we recall that the vacuum in the matrix model is described by block-diagonal matrices with blocks given in (2.37)-(2.38), corresponding to  $SU(2)$  irreducible representations of dimension  $J_{(1)}/k$  and

<sup>22</sup>We use the symbol  $O(1)$  to signify that the quantity in question is much smaller than  $J/2k$ .

$J_{(2)}/k$ , with  $J_{(1)} + J_{(2)} = J$ . When considering fluctuations around these configurations one turns on entries in the entire matrices, including the off-diagonal blocks, which correspond to  $(J_{(1)}/k) \times (J_{(2)}/k)$  rectangular matrices. These rectangular matrices are the natural candidates to describe excitations dual to the slow modes associated with the  $(1, 2)$  and  $(2, 1)$  components of the ABJM fields. The corresponding spectrum was computed in [24] using a Clebsch-Gordan method. The basis used in the computation of the spectrum of fluctuations in the rectangular off-diagonal blocks was further systematically studied in [84], where a direct correspondence between this basis and the monopole spherical harmonics was pointed out. More specifically in [84] it was shown that rectangular  $(J_{(1)}/k) \times (J_{(2)}/k)$  matrices can be expanded in a basis consisting of a discretised version of the monopole spherical harmonics with charge  $q_{(1)} - q_{(2)} = (J_{(1)} - J_{(2)})/2k$ . The scalar fluctuations coming from  $S^7$  directions are  $(X^n)^u_v$ , where  $n = 4, \dots, 9$  and the matrix indices,  $u$  and  $v$ , span the off-diagonal (rectangular) blocks. Their energies are given by [24]

$$\omega = \frac{2}{R} \left( \frac{1}{2} + l \right), \quad (4.4)$$

where the quantum number  $l$  takes values

$$\frac{1}{2k} |J_{(1)} - J_{(2)}| \leq l \leq \frac{1}{2k} (J_{(1)} + J_{(2)}) - 1. \quad (4.5)$$

There are six polarisations, corresponding to  $n = 4, \dots, 9$ , hence for each  $l$  in the range (4.5) the multiplicity is  $6 \times (2l + 1)$ .

Using  $\omega = (2\Delta - J)/R$  and  $q_{(i)} = J_{(i)}/2k$ ,  $i = 1, 2$ , the matrix model spectrum (4.4)-(4.5) agrees with the result (4.3) for the slow modes associated with the  $(1, 2)$  and  $(2, 1)$  components of the scalars  $\phi^{A'}$  in the ABJM theory, verifying the AdS/CFT duality for this particular set of states.

Notice that in the matrix model there is a built-in upper bound in the range (4.5) for the quantum number  $l$ . As remarked at the end of section 2.3 and in section 3.3 after (3.79), in view of the approximation schemes that we are using on the two sides of the correspondence, we can only expect good quantitative agreement for low-lying states in the spectra, with quantum number  $l \ll J/2k$ . Hence the absence of a corresponding upper bound on the CFT side would not necessarily lead to a contradiction. However, the Born-Oppenheimer scheme does indeed suggest the existence of a similar upper bound, as it is natural not to consider modes with large  $l$  – and in particular  $l \gtrsim J/2k = (J_{(1)} + J_{(2)})/2k$  – as slow modes <sup>23</sup>.

The agreement between the spectra of these low-energy off-diagonal modes has interesting implications. The off-diagonal blocks in the regularised multi-membrane sectors have no obvious interpretation in the conventional continuum membrane theory. Thus the matrix model contains additional degrees of freedom with no counterpart in the membrane theory. The fact that, at least when  $|J_{(i)} - J_{(j)}| \ll J$ , these fluctuations have corresponding low-energy states in the ABJM theory – and the spectra on the two sides match – indicates that these are genuine M-theory degrees of freedom and not an artefact of the matrix regularisation. Therefore our results provide an explicit and concrete example showing that the matrix model can capture aspects of the dynamics of M-theory beyond the conventional supermembrane

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<sup>23</sup> The numerical coefficient in the expression for the cut-off should not be taken too seriously. As is always the case with low-energy effective descriptions, the significance of such a bound is only in setting a separation between states with quantum numbers much below and much above a certain value.

theory [85]. The existence of the extra degrees of freedom, appearing when the two membranes are close to each other (as the condition  $|J_{(i)} - J_{(j)}| \ll J$  implies), can be thought of as a manifestation of the non-Abelian nature of membranes, analogous to that of D-branes <sup>24</sup>. It would be interesting to verify that the agreement discussed above between the energies of these particular states on the two sides of the AdS/CFT duality persists after the inclusion of quantum corrections. We hope to investigate this issue in the future.

## 5 Conclusions and Discussion

In this paper we have studied the  $\text{AdS}_4/\text{CFT}_3$  duality proposed in [13] in an M-theoretic regime in which neither the ten-dimensional type IIA string limit nor the low-energy eleven-dimensional supergravity approximation are applicable. In order to make it possible to quantitatively study the correspondence in this regime, we have focussed on a special sector associated with a large quantum number,  $J$ . On the gravity side  $J$  is an orbital angular momentum and for large  $J$  the membrane configurations we consider can be described using the pp-wave matrix model. On the CFT side the dual sector involves monopole operators which are conveniently studied using a Hamiltonian formulation within the framework of radial quantisation. In this approach we consider states satisfying a Gauss law constraint associated with the presence of a large flux, controlled by the parameter  $J$ , through the  $S^2$  corresponding to fixed-time slices in radial quantisation. In the large  $J$  regime we identified approximation schemes which are simultaneously valid on both sides of the duality. On the one hand the pp-wave matrix model is weakly coupled and therefore a standard quantum mechanical perturbative expansion is applicable. On the other hand in the ABJM theory the presence of a large parameter makes it possible to give a (weakly coupled) effective description of the physical degrees of freedom dual to M-theory states using a Born-Oppenheimer approach. The choice of a suitable gauge is a crucial element of our analysis on the CFT side. Another essential ingredient is a version of the Higgs mechanism, which, together with the presence of a large magnetic flux, leads to a separation between low and high energy states thus allowing us to identify the physical degrees of freedom.

When using radial quantisation and the state-operator map, the AdS/CFT dictionary directly relates energy spectra on the two sides of the duality. We have verified the agreement between these spectra in the large  $J$  sector at leading order for both BPS and near-BPS states. This provides a very non-trivial test of the  $\text{AdS}_4/\text{CFT}_3$  duality of [13] in an M-theoretic regime which had not been accessible so far. At the same time, by independently reproducing the membrane spectrum from the dual CFT, our results provide strong support for the validity of the matrix model approach to M-theory.

The AdS/CFT dictionary we proposed is summarised in table 5.

The starting point of our analysis – *i.e.* the observation that focussing on a sector characterised by a large quantum number leads to a simplification in the study of the AdS/CFT

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<sup>24</sup> This non-Abelian character is manifest in the Bagger-Lambert and ABJM theories [13–15], which were proposed as low-energy descriptions of multiple membranes. The possibility of interpreting the block off-diagonal components in the pp-wave matrix model as the non-Abelian degrees of freedom of membranes was suggested in [24]. In [47] it was pointed out that the non-Abelian nature of membranes may explain certain interesting properties of stable solutions (corresponding to membranes with torus topology) in a deformed version of the matrix model, where configurations of membranes characterised by different winding numbers in the continuum theory become indistinguishable in the matrix model description.

	AdS side	CFT side
Framework	pp-wave matrix model	Radial quantisation with large flux
Approximation	pp-wave approximation and loop expansion	Born-Oppenheimer
BPS ground states	Collection of fuzzy spheres	Flux characterised by GNO charges
Near BPS fluctuations	6 real scalars from $S^7$ 3 real scalars from $\text{AdS}_4$ 16 real fermions	3 complex scalars $\phi^{A'}$ ( $A' = 1, 2, 3$ ) $\phi^4$ and gauge fields 4 complex spinors $\psi^A$

Table 5: Dictionary for M-theoretic  $\text{AdS}_4/\text{CFT}_3$ . The pp-wave matrix model Hamiltonian is (2.26). The Hamiltonian of ABJM theory in our gauge, which incorporates the effect of magnetic flux, is (3.62). The BPS ground states are classified on both sides of the duality by a partition of integers satisfying (2.39) on the AdS side and (3.19) on the CFT side. The near-BPS fluctuation spectrum on the AdS side is summarised in table 1. On the CFT side, the corresponding degrees of freedom are (1,1) component of various fields and their spectrum is given in table 3.

duality – is similar to the premise of the work of BMN in the context of the  $\text{AdS}_5/\text{CFT}_4$  correspondence [23]. More generally there are analogies between our construction and that of [23]. However, the final picture that emerges from our investigation is fundamentally different from the one proposed by BMN. This is the manifestation of the fact that we have applied similar ideas to the description of a very different physical system – membranes rather than strings.

The relationship between the AdS and CFT sides of the duality we have studied in this paper seems to be remarkable for its directness. In particular, in comparing the two sides of the correspondence the sphere introduced on the CFT side as a tool in the radial quantisation can almost be identified with the sphere representing the minimal energy configuration for membranes on the matrix model side. The implication of this observation is that the states on the two sides are naturally described in terms of the same (monopole) spherical harmonics, making the definition of the map between bulk and boundary observables more straightforward. This may not be so surprising since the important degrees of freedom in the bulk of  $\text{AdS}_4 \times S^7/\mathbb{Z}_k$  are membranes and the boundary ABJM theory describes the low-energy dynamics of membranes, so that on both sides one focusses on the same kind of objects. This is in strong contrast with more familiar examples of AdS/CFT duality and in particular the canonical  $\text{AdS}_5/\text{CFT}_4$  case, where the bulk degrees of freedom are closed fundamental strings and the boundary theory describes the low-energy degrees of freedom of D3-branes.

It is essential that we use large but finite  $J/k \times J/k$  matrices on the AdS side. This is in particular crucial in establishing the map relating BPS states on the two sides of the duality, which are classified by a set of integers – associated respectively with the angular momenta of individual membranes in the matrix model and with GNO charges of monopole operators in the CFT. The fact that we can formulate a duality with the ABJM theory using finite dimensional matrices is interesting. In this respect our construction is different from the standard approach to the matrix model description of membranes [3, 4], in which the size of the matrices plays the role of a regularisation parameter and should be taken to infinity. The matrix model seems to describe a theory in which the membranes are discretised. This is reminiscent of the description of gauge invariant operators dual to closed strings in terms of a discrete spin chain in versions of the AdS/CFT correspondence in which the gravity dual is a string theory. A consequence of working with matrices of finite size is the presence of an upper bound on the mode numbers in the expansion of the fluctuations in spherical harmonics. We have seen that a corresponding cut-off naturally arises on the CFT side in the context of the Born-Oppenheimer approximation. Here it follows from the fact that it is not completely justified to treat as low-energy modes the high momentum components of the slow modes with energies higher than the mass of the fast modes.

The implications of the direct nature of the duality we have presented are particularly intriguing in the case studied in section 4, where there are multiple concentric spherical membranes of approximately equal radius. In this situation we have seen that the block off-diagonal degrees of freedom of the matrix model have as counterpart in the dual ABJM theory certain off-diagonal components of the fields. The block off-diagonal degrees of freedom do not exist in the conventional continuum membrane theory, which does not take into account the possibility that membranes possess non-Abelian degrees of freedom. The fact that their spectrum appears to be reproducible in the CFT suggests that these degrees of freedom should not be considered as spurious, or a kind of “lattice artefact”. Instead they seem to be the manifestation of a genuinely non-Abelian nature of membranes in M-theory. This is a new and non-trivial insight into the dynamics of M-theory that can be deduced from the study of the AdS/CFT correspondence. It would be also interesting to understand this non-Abelian nature of membranes directly from the matrix model without relying on the AdS/CFT correspondence. This presumably will help to shed light on a possible non-Abelian Born-Infeld type description of membranes.

Another interesting feature of the ABJM theory which emerges from our analysis is the following. Let us consider the case in which only one membrane is present on the AdS side and correspondingly only the first GNO charge is non-zero on the CFT side. In this case, as we have seen in section 3.2, the excitation of the  $(1, 1)$  components of the ABJM fields is identified with the excitation of phonons on the stable spherical membrane. On the other hand, the excitations of (diagonal) components in the lower right  $(N - 1) \times (N - 1)$  block would in general give rise to other non-zero GNO charges. We should interpret this as the creation of additional membranes <sup>25</sup>. Thus the ABJM theory combines the features of a first-quantised and a second-quantised description of membranes in this manner.

At the end of section 3.3 we have presented the estimate (3.92) for the behaviour of loop corrections in the ABJM theory in terms of powers of  $N$ . However, an intriguing possibility

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<sup>25</sup>Strictly speaking, this argument is partially based on an extrapolation of the results of our analysis valid for  $J \gg 1$ , as the momenta/GNO charges of the created membranes may not be large.



is to retain the  $(N - 1)$  combination, which has an interesting explanation in terms of the gravity dual. In our interpretation of the ABJM theory in the large  $J$  sector the slow modes correspond to the membrane fluctuations studied in section 2. The  $\text{AdS}_4 \times S^7/\mathbb{Z}_k$  background is obtained as the near horizon geometry of a stack of  $N$  membranes. One can think of the fluctuations as coming from excitations of the original  $N$  background membranes. Then for a state containing a single excited membrane the background comprises only the remaining  $(N - 1)$  membranes. It may be possible to interpret the  $(N - 1)$  fast modes as corresponding to the background associated with these  $(N - 1)$  membranes. This picture is also consistently generalised to the case of multi-membrane configurations on the gravity side, which is related to a sector of the CFT with multiple non-zero GNO charges. For instance in the case of two membranes/GNO charges briefly discussed in section 4 – at least if the two membranes have comparable angular momenta – the fast modes can be combined into groups of  $(N - 2)$  fields. This corresponds to the fact that on the AdS side we have two excited states leaving a background of  $(N - 2)$  membranes.

The general analysis of perturbative corrections in section 3.3 suggests the possibility of the emergence of a novel type of large  $N$  expansion in the ABJM theory for  $J \gg 1$ . We have provided a prescription for determining the dependence on powers of  $(N - 1)$  in the single membrane sector. This involves drawing Feynman diagrams in the standard double line notation and then specifying for all index loops whether they carry a colour index 1 or a primed index taking values  $2, \dots, N$ . The different perturbative contributions can be classified according to the power of  $(N - 1)$  they produce. This power is given by the number of index loops carrying primed colour indices. The resulting large  $N$  expansion is different from the standard 't Hooft expansion. Moreover, the general considerations on the structure of the diagrammatic corrections to the spectrum discussed in section 3.3 – and specifically the expected cancellations due to supersymmetry – suggest a relation between the order in the loop expansion and the powers of  $(N - 1)$ , which is inherently new. It is well-known that if one focusses on the contribution of planar diagrams in the 't Hooft expansion, a theory often simplifies and shows various special properties. It would be interesting to study whether the leading order contributions in this new type of large  $N$  expansion have similar special properties. It is intriguing to speculate that the emergence of this new type of large  $N$  expansion may be related to the fact that we are considering a genuinely M-theoretic regime. In the sector under consideration the elementary degrees of freedom on the gravity side are not strings, whereas the standard 't Hooft expansion suggests strongly a stringy interpretation for the fundamental degrees of freedom.

The most important next step in our program will be to compute the higher order corrections in the spirit of the Born-Oppenheimer approximation discussed in section 3.3. The calculation is quite involved as the gauge-fixed Hamiltonian, obtained iteratively solving the Gauss law constraints, contains a very large number of interaction vertices. Therefore it will be crucial to develop techniques to simplify the computations.

Although the gauge adopted in this paper seems to be well-suited to clarify the structure of the physical degrees of freedom, there may be more convenient choices for explicit loop computations. The situation may be analogous to the well-known case of Yang-Mills theories when the gauge symmetry is spontaneously broken via the Higgs mechanism. In that case, the unitary gauge is well-suited for studying the spectrum of the theory, but there are other gauge choices which are more convenient to perform loop computations.



The spectrum of the slow modes and the effective theory governing their dynamics discussed in section 3, should be highly constrained by supersymmetry. It is important to concretely study the restrictions imposed by supersymmetry. This should also facilitate the explicit computation of quantum corrections. As discussed in section 3.3, the results on the AdS side of the correspondence indicate the presence of cancellations leading to certain patterns in the structure of higher order perturbative corrections in the CFT. It will be important to explore this structure directly in the ABJM theory by carrying out calculations of loop corrections to the spectrum.

An efficient computational scheme may arise from the adaptation of the methods based on localisation to the study of the sector we focussed on. The localisation approach relies on the existence of a nilpotent supercharge which annihilates the observables under consideration. As such the method is only applicable to BPS quantities which are invariant under at least one supersymmetry. To implement the method one deforms the ABJM theory by the addition of terms invariant under the relevant supercharge. The deformation is controlled by a parameter in such a way that when the parameter is sent to infinity a saddle point approximation becomes exact, allowing an explicit evaluation of the observables. The main focus of our investigation are non-BPS states and for this reason in order to study their spectrum we have relied on a different approximation scheme that arises for large  $J$ . However, the observables we have considered – similarly to the BMN operators in the  $\mathcal{N} = 4$  supersymmetric Yang-Mills theory – are near BPS, with  $1/J$  acting as the parameter measuring their deviation from exactly BPS observables. In view of this it might be possible to generalise the localisation approach to make it applicable to near-BPS observables. In this case the deformation parameter cannot be sent strictly to infinity, because the observables are not invariant under the deformation. However, for a carefully chosen deformation, we expect the variation of near BPS observables such as those studied in this paper to be suppressed by a power of  $1/J$ . It should then be possible to take the deformation parameter to be as large as a positive power of  $J$  and evaluate near BPS observables using a saddle point approximation, including higher order terms in an expansion in  $1/J$ . This approach may potentially have many applications beyond the large  $J$  sector of the ABJM theory and it would be interesting to test this idea in simple models.

The sector we have discussed provides a setting for the study of the interactions responsible for processes involving the splitting or joining of membranes. This is a central and still little understood aspect of the dynamics of M-theory and the pp-wave matrix model, together with its dual description in terms of the ABJM theory, appears to be particularly suited to investigate it. The perturbative vacua on the gravity side, which correspond to configurations with varying numbers of membranes, actually belong to the same Hilbert space of the matrix model. In other words, as is well-known, the matrix model should be interpreted as a second-quantised theory of membranes [6, 46]. It should be possible to compute transition amplitudes between two states (either “vacua” or excited states) characterised by different sets of integers as already mentioned in section 2.1. These transition amplitudes are analogous to the string field theory vertices in the ten-dimensional pp-wave background and thus should provide the building blocks for the computation, from the dual gravitational perspective, of  $n$ -point correlation functions of the operators we defined via the state-operator map. Analogous computations for the three-point functions of BMN operators have been done in string theory. For a recent reference, see [86]. For an analysis of the relation between the

transition amplitudes (or the string vertex) and the CFT OPE coefficients, see [36]. It is important to determine the coupling constant governing these processes, which should correspond to tunnelling amplitudes. In this paper we have assumed this coupling to be small. It is tempting to conjecture that it may be given by a certain combination of powers of  $N$ ,  $k$  and  $J$ . It is also an interesting problem to compute three-point functions of operators with non-zero monopole charges, such as those considered in this paper, directly on the CFT side. This is presumably related to the tunnelling process discussed at the end of section 3.1.

In this paper we studied the M-theory regime in which the parameter  $k$  is of order 1. It is of some interest to consider whether there is a type IIA regime ( $N \gg 1, k \gg 1$  with  $N/k$  fixed) in which a description similar to the one given in this paper based on the pp-wave approximation is possible. An essential difference in the type IIA regime is that, since the M-theory circle is small, M2-branes wrapped on the M-theory circle should also be considered. To incorporate these degrees of freedom it seems appropriate to use the matrix string formulation [87, 88]. Several works have studied aspects which are relevant for this line of investigation. A direct map identifying the degrees of freedom associated with wrapped membranes in matrix string theory was discussed in [89]. The matrix string theory on a type IIA supersymmetric pp-wave background was constructed in [90, 91]. An M2-brane solution in the type IIA regime, which is wrapped around the M-theory circle and has torus topology, was found in [92]. Wrapped M5-brane solutions related to the wrapped M2-brane solutions were discussed in [93]. Considerations on the CFT counterparts of these solutions were presented in [60, 94]. There may be connections to BPS solutions of the membrane theory on the pp-wave background with arbitrary genus found in [95]. We also note that the pp-wave approximation for string states in the type IIA limit with zero monopole charge was studied in [96, 97].

In recent years methods derived from the study of integrable systems have played an important role in the computation of corrections to the spectrum on both sides of the AdS/CFT correspondence [98]. A question that arises is whether integrability can be relevant in the sector of the  $\text{AdS}_4/\text{CFT}_3$  duality that we considered in this paper. The consensus is that integrability in the AdS/CFT correspondence is a feature arising only in the planar approximation. This seems to indicate that it should not be expected in the M-theoretic regime. However, since we are dealing with a theory of membranes rather than strings, the significance of the planar approximation is unclear. An interesting possibility is that integrability might arise in the large  $J$  sector of the ABJM theory if one focusses on the leading contributions in the novel large  $N$  expansion that we described above. Moreover, extending the ideas developed in the context of stringy examples of AdS/CFT duality, it is natural to expect that the relevant integrable systems in a case involving membrane degrees of freedom might be  $2 + 1$  dimensional. These considerations lead us to suspect that, if integrability can play a role in the present context, it should present interesting new features.

As already noted above constraints from supersymmetry will presumably play a crucial role in better understanding the structure of the ABJM theory in the large  $J$  regime. For  $k = 1, 2$  supersymmetry is expected to be enhanced to  $\mathcal{N} = 8$ . The extra supersymmetries are related to the presence of monopole operators and some of the associated R-currents are already known [56, 57]. More concretely, the part of the  $\mathcal{N} = 8$  supersymmetry algebra broken for  $k \neq 1, 2$  corresponds to generators transforming under  $J_M$  with charge  $\pm 2$ . Since the monopole charge,  $J_M$ , is a multiple of  $k$ , these charges cannot exist for  $k \geq 3$  and

this explains why the  $\mathcal{N} = 8$  supersymmetry is broken down to  $\mathcal{N} = 6$  for  $k \geq 3$ . In section 3.2 we have focused on states with given  $J_M$  in the gauge-fixed theory. It seems to be straightforward to relax this restriction. Since  $J_M$  is a conserved charge, the Hilbert space of the ABJM theory can be viewed as the direct sum of the vector spaces of states with fixed  $J_M$ . The action of the full superalgebra, including the supercharges that change the value of  $J_M$  by  $\pm 2$  units, spans the entire Hilbert space of the theory relating states with different quantum number  $J_M$ . Since our gauge has the advantage of being very explicit, it should be possible to use it to write down the supercharges for the full  $\mathcal{N} = 8$  supersymmetry, at least at the classical level. It would be interesting to do this and study their commutators to explicitly verify the closure of the  $\mathcal{N} = 8$  superalgebra.

The mechanism of the breaking of  $\mathcal{N} = 8$  supersymmetry explained above helps to clarify why the pp-wave matrix model has 32 real supersymmetries [23, 25]. At leading order in the pp-wave approximation order 1 differences in the value of  $J_M \sim J$  cannot be detected. Hence it is natural to expect that the  $\mathcal{N} = 8$  supersymmetry of the matrix model for arbitrary  $k$  should be interpreted as an approximate symmetry, which would be broken by the inclusion of corrections to the pp-wave approximation. In a similar fashion the low-energy sector that we identified in the ABJM theory for large  $J$  should possess an approximate  $\mathcal{N} = 8$  supersymmetry even for  $k \geq 3$ . It would be interesting to study concretely these aspects, analysing the corrections to the pp-wave approximation on the gravity side and the symmetries of the low-energy effective theory for the slow modes on the CFT side.

Following the ABJM proposal there have been many generalisations leading to other examples of  $\text{AdS}_4/\text{CFT}_3$  dualities with less supersymmetry. It should be possible to extend our analysis to these cases as well. The approach and the techniques we have developed in this paper will also be useful more generally in the study of various properties of three-dimensional conformal Chern-Simons-matter theories, irrespective of whether or not they have gravity duals. More specifically, the Born-Oppenheimer type approximation we have discussed in section 3 provides a new approach to the computation of conformal dimensions of various types of operators with large monopole charge, which may be applicable in situations where a conventional perturbative expansion is not justified.

The work presented in this paper has interesting connections to the little understood  $(2, 0)$  superconformal field theory in six dimensions. This theory, which is believed to describe the low-energy dynamics of a stack of M5-branes, is expected to have as gravity dual M-theory in an  $\text{AdS}_7 \times S^4$  background. It is interesting to notice that the application of the pp-wave approximation to this background leads to the same geometry as the one obtained from  $\text{AdS}_4 \times S^7$  [23, 34]. This suggests that the six-dimensional  $(2, 0)$  theory should contain a sector dual to the matrix model we have considered for which a weak coupling description might be possible. Since M2- and M5-branes are electromagnetic duals of each other in the eleven dimensional target space of M-theory, one may expect the M2-brane excitations discussed in section 2.2 to be captured by solitonic degrees of freedom in the  $(2, 0)$  theory.

According to the proposal of [48] there are states in the pp-wave matrix model which have dual descriptions as M5- or M2-branes. For instance a configuration characterised by a partition  $J = 1 + 1 + \dots + 1$  can be identified with a single M5-brane. The pp-wave approximation should be applicable to such M5-branes as discussed in section 2, since they carry large angular momentum and their size is small. Even if they cannot be treated perturbatively in the matrix model, it is possible that they can be studied by a Monte-Carlo simulation or

by devising an appropriate approximation scheme such as a variational method. It would be interesting if one could gain any insights into the dynamics of M5-branes using the pp-wave matrix model and its dual description in terms of the ABJM theory.

We have seen that the large  $J$  limit seems to provide a good framework in which concepts from M-theory – and its matrix formulation – and the AdS/CFT duality work together. We hope that the interplay of these ideas may lead to a better understanding of both M-theory and the AdS/CFT correspondence.

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